

MODELING OF RADIATION WAKEFIELDS USING A SCATTERED-FIELD FORMULATION

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Abstract

Predicting Coherent Synchrotron Radiation (CSR) generated by bent trajectories in electron accelerators imposes significant challenges for analytical modeling and numerical simulations. We present a modeling approach targeting indirect CSR effects created by reflections at chamber walls. Unlike conventional CSR models, the method accounts for arbitrary geometries and bunch dynamics, including dispersion and compression. Computational efficiency is maintained through a scattered-field formulation, which permits the simulation of reflected CSR wakes directly without resolving the internal bunch structure in the same grid. To smooth the radiated field of the particles, point particles are distributed along their trajectory using a temporal variant of the continuous-charge approach. We demonstrate the method for the BC0 bunch compressor of the European XFEL.

INTRODUCTION

The generation of Coherent Synchrotron Radiation (CSR) by electrons undergoing acceleration remains a major source of beam quality degradation in accelerator sections [1]. Especially for electron bunches moving on curved trajectories, CSR can cause energy modulation, emittance growth and microbunching. However, prediction of beam dynamics under the influence of CSR effects remains a challenging task for both analytical and numerical approaches [2].

Analytical CSR models often start from steady-state motion, i.e., a rigid bunch on a circular path. The models are refined for some pre-defined trajectories, with the straight-bent-straight path caused by a finite-length dipole being the most prominent example [3, 4]. To additionally account for shielding effects caused by surrounding geometry, extensions have been developed for parallel plates and, with additional restrictions, for rectangular chambers [4–6]. Numerical approaches using a particle-in-cell (PIC) method are, despite their generality, infeasible for most cases of interest due to the required fine resolution in both space and time. Hence, CSR effects are often accounted for by incorporating (semi-)analytical models in particle tracking codes, inheriting their underlying assumptions [3, 7]. For such models, satisfying agreement between measurements and simulations was achieved [1]. In contrast, some codes solve the Liénard-Wiechert (LW) potentials directly to simulate arbitrary ensembles and trajectories in free space. However, even with advanced schemes and high-performance computing, the computational effort remains high due to the required particle number and the single-particle based evaluation [8, 9].

These numerical approaches focus on the direct part of the CSR field, with the effect of surrounding geometry included in a simplified manner, if at all. In contrast, a mesh-based full-wave electro-magnetic field solver can account for arbitrary geometry but models the electron bunch as current excitation, which neglects e.g. compression [10].

To close this gap, particularly for low- to mid energy sections, we present a method to simulate the indirect (scattered) part of the CSR fields while allowing for arbitrary beam trajectories and geometries. We extend our previous approach [11] by retardation and radiation effects in combination with a variant of the continuous-charge approach by Dohlus [12] to smooth the particle far field.

METHODOLOGY

Scattered-Field Formulation

We briefly summarize the scattered-field formulation developed for the coupled simulation of wakefield and space charge problems [11]. Maxwell's wave equations govern this problem, subject to perfectly electrically conducting (PEC) boundary conditions at the interface Γ and driven by the particle beam current density $\mathbf{J}$,

$$\partial_t \mathbf{D} - \nabla \times \mathbf{H} = -\mathbf{J}, \quad (1a)$$

$$\partial_t \mathbf{B} + \nabla \times \mathbf{E} = 0, \quad (1b)$$

$$\mathbf{n} \times \mathbf{E}|_{\Gamma} = 0 \quad (1c)$$

with electric and magnetic field strengths, $\mathbf{E}$ and $\mathbf{H}$, as well as their corresponding flux densities, $\mathbf{D}$ and $\mathbf{B}$. These are related by the constitutive relations in vacuum, $\mathbf{D} = \epsilon_0 \mathbf{E}$ and $\mathbf{B} = \mu_0 \mathbf{H}$. The normal vector to Γ is denoted $\mathbf{n}$. The scattered-field formulation decomposes this *total* field $\mathbf{E}$ into an *incident* $\mathbf{E}_i$ and a *scattered* $\mathbf{E}_s$ component. The incident field fulfills (1a) and (1b), however with free-space boundary conditions instead of (1c). The scattered field, on the other hand, fulfills (1b) and the homogeneous version of (1a), with its excitation arising solely from correcting the unphysical incident field at the PEC boundary. In summary, the scattered-field formulation reads

$$\mathbf{E} = \mathbf{E}_i + \mathbf{E}_s \quad (2a)$$

$$\mathbf{n} \times \mathbf{E}_s|_{\Gamma} = -\mathbf{n} \times \mathbf{E}_i|_{\Gamma}. \quad (2b)$$

This decomposition allows to separate scales and effects, and to use different methods for the respective fields, at the cost of a new boundary condition for the scattered field. Hence, we realize the scattered-field formulation (2) by coupling our in-house space-charge solver REPTIL [13] for the incident field with our wakefield solver PBCI [14] for the scattered

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field. The latter is extended by the boundary condition (2b). So far, the incident field at the boundary $\mathbf{E}_i|_\Gamma$ was evaluated in the Lorentz rest frame of the bunch, an approximation that is well suited for predominantly linear motion such as inside the electron gun, yet insufficient for CSR effects.

Temporal Continuous-Charge Approach

We recall that the electromagnetic field created by a point particle of charge q moving along the trajectory $\mathbf{r}_p = \mathbf{r}_p(t)$ in free space is given by the LW field equations [15],

$$\mathbf{E}_{\text{LW}}(\mathbf{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\frac{\mathbf{n}_p - \boldsymbol{\beta}_p}{\gamma_p^2 |\mathbf{r} - \mathbf{r}_p|^2} + q \frac{\mathbf{n}_p \times [(\mathbf{n}_p - \boldsymbol{\beta}_p) \times \dot{\boldsymbol{\beta}}_p]}{c |\mathbf{r} - \mathbf{r}_p|^3}}{4\pi\epsilon_0 (1 - \mathbf{n}_p \cdot \boldsymbol{\beta}_p)^3} \bigg|_{t=t_r} \quad (3)$$

$$\mathbf{B}_{\text{LW}}(\mathbf{r}, t) = \frac{1}{c} \mathbf{n}_p \times \mathbf{E}_{\text{LW}}(\mathbf{r}, t),$$

where $\mathbf{n}_p = (\mathbf{r} - \mathbf{r}_p)/|\mathbf{r} - \mathbf{r}_p|$, $\boldsymbol{\beta}_p = \dot{\mathbf{r}}_p/c$ and $\gamma_p = (1 - \boldsymbol{\beta}_p^2)^{-1/2}$ is the Lorentz factor. All p -quantities are evaluated at the retarded time t_r defined implicitly via

$$c(t - t_r) = |\mathbf{r} - \mathbf{r}_p(t_r)|. \quad (4)$$

In principle, (3) can be used as incident field $\mathbf{E}_i|_\Gamma$ for the excitation (2b). However, the far field created by a set of point particles produces unbearable shot noise for the wakefield solver, as will be shown later. One alternative is the continuous-charge (CC) approach proposed in [12], which derives the equivalent of (3) for a particle with a spatial 3D Gaussian charge distribution. The resulting expressions still require the numerical evaluation of a 1D integral. We propose a variant of this approach by continuously distributing a single particle along its trajectory, i.e. in time only. This idea is depicted in Fig. 1.

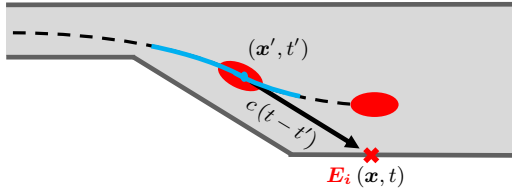


Figure 1: Illustration of the temporal continuous-charge approach to evaluate the radiation field of a particle on the boundary, using a continuous charge distribution (blue) along the trajectory (black, dotted) instead of a single retarded position $\mathbf{x}'$.

A single particle is assigned a temporal density function $n(\tau)$, where τ is the time shift along the reference trajectory $\mathbf{r}_p$. Then, the charge density reads

$$\rho(\mathbf{r}, t) = q \int_{-\infty}^{\infty} \delta(\mathbf{r} - \mathbf{r}_p(t - \tau)) n(\tau) d\tau. \quad (5)$$

As the LW fields (3) are derived for the point charge $n(\tau) = \delta(\tau)$, the fields of the continuous-charge distribution $\mathbf{E}_{\text{cc}}$ are obtained by convolution with the field kernel $\mathbf{E}_{\text{LW}}$,

$$\mathbf{E}_{\text{cc}}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \mathbf{E}_{\text{LW}}(\mathbf{r}, t - \tau) n(\tau) d\tau, \quad (6)$$

with an analogous expression for $\mathbf{B}_{\text{cc}}$. Using (4) enables us to rewrite (6) with respect to the retarded time t_r ,

$$\mathbf{E}_{\text{cc}}(\mathbf{r}, t) = \int_{-\infty}^{\infty} \mathbf{E}_{\text{LW}}(\mathbf{r}, t_r) n\left(t - t_r - \frac{|\mathbf{r} - \mathbf{r}_p|}{c}\right) dt_r. \quad (7)$$

Furthermore, we note that during temporal discretization in the wakefield solver, the degrees of freedom at time instant t^m are understood as averages over the corresponding time interval. This averaging is incorporated by an additional convolution with a window function,

$$\begin{aligned} \langle \mathbf{E}_{\text{cc}} \rangle(t^m) &= \frac{1}{\Delta t} \int_{t^m - \Delta t/2}^{t^m + \Delta t/2} \mathbf{E}_{\text{cc}}(t) dt \\ &= \int_{-\infty}^{\infty} \mathbf{E}_{\text{LW}}(t_r) \times \\ &\quad \underbrace{\frac{1}{\Delta t} \int_{t^m - \Delta t/2}^{t^m + \Delta t/2} n\left(t - t_r - \frac{|\mathbf{r} - \mathbf{r}_p|}{c}\right) dt}_{=: N_m(t_r)} dt_r \end{aligned} \quad (8)$$

where we drop the spatial dependence for brevity and introduce the (windowed) density function $N_m(\tau)$. Note, that changing from a point particle to a smooth distribution n or N acts as a low-pass filter on the radiated field, as pointed out e.g. by [16]. Although this filter would be disadvantageous for the simulation of the direct part of the CSR field by suppressing microbunching effects [17], it is beneficial for the (collective) scattered field. In practice, evaluation of (7) and (8) starts by finding the retarded time for the reference point particle, followed by stepping forward and backward along the trajectory while accumulating the corresponding LW fields weighted by n or N_m . Using small weights as breakup criterion, this procedure is inherently adaptive and, outside the light cone, rapidly converges to the point-particle case.

RESULTS

To demonstrate the properties of the proposed continuous-charge approach, we apply it to the simulation of the BC0 bunch compressor of the European XFEL. Located at the end of the injector section, it reduces the bunch length from around 3 mm to 1 mm at an energy of 130 MeV [18]. The geometric details of the chamber can be found in [10, 19], a cross-sectional view is given in Fig. 2.

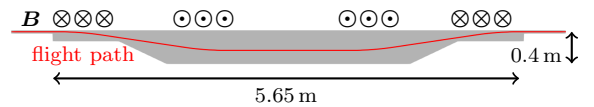


Figure 2: Cross-sectional view of the vacuum chamber of the BC0. The reference trajectory (red) and the dipole fields are indicated.

First, we evaluate the far field along a straight observation line parallel to the trajectory downstream from the first bend.

Figure 3 shows this field created by a bunch of particles for one time instant. Point-particle evaluation (LW) is compared to the temporal continuous-charge (CC) approach with a Gaussian $n(\tau)$. The standard deviation of the Gaussian is chosen such that adjacent macroparticles overlap slightly. Inside the light cone of the bunch – the region with high field strength – the sampling of the line is insufficient for the point-particle LW fields and causes aliasing artifacts, even for a large number of macroparticles. In contrast, the CC yields a noise-free result for a comparatively low number of macroparticles.

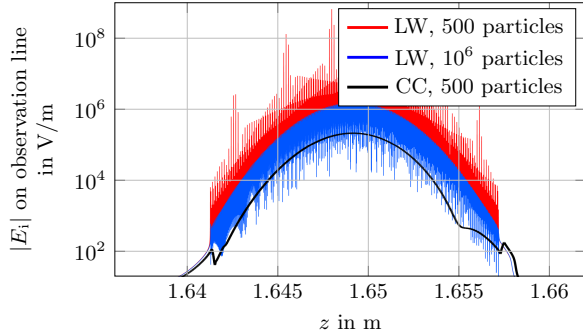


Figure 3: Evaluation of the incident field on an observation line in the chamber.

To compare our temporal continuous-charge approach to the 3D spatial one in [12], we evaluate the field of a single particle along the same observation line. For sufficiently small and matching standard deviations in the corresponding Gaussian charge distributions, the same far field is expected. The result is shown in Fig. 4 and shows excellent agreement in the direct vicinity of the light cone. However, resolving the spatial CC outside the light cone requires increasingly higher temporal resolution of the trajectory. Violation of this requirement leads to the nonphysical results observed in the left part of the plot, whereas the temporal CC still yields a smooth field.

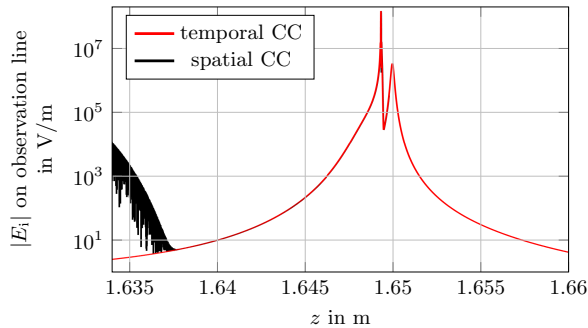


Figure 4: Comparison of the continuous-charge approach by [12] (black) and our variant (red) for the single-particle far field. Gaussian distributions with the same standard deviation are used in both cases.

In the following, we simulate the whole BC0 section using the scattered-field formulation with the temporal continuous-charge approach for the incident field evaluation. The initial electron bunch is distributed on-axis and sampled with 500

macroparticles as well as a series of test particles on the reference trajectory. Each particle is represented by a windowed Gaussian profile N . Internal space charge forces are neglected to focus on the indirect CSR wakefield. Simulations are carried out with and without compression, i.e. with and without initial energy chirp. Without compression, a fixed bunch length of $\sigma_z = 2$ mm is used. With compression, we choose a reduction in σ_z from 3 mm to 1.5 mm. The longitudinal wake potentials are compared in Fig. 5. For the compression case, results are shown separately for macroparticles of the bunch following the dispersive paths (green) and the test particles on the reference trajectory (blue). Inside the bunch, both cases exhibit similar behavior, while the oscillations in the wake tail are more pronounced for the compression case. The high-frequency oscillations in the tail indicate that a finer resolution would be required, at the expense of increased computational cost. A wake building up ahead of the bunch is also observed, a characteristic feature of CSR wakes overtaking the particles along the curved trajectory.

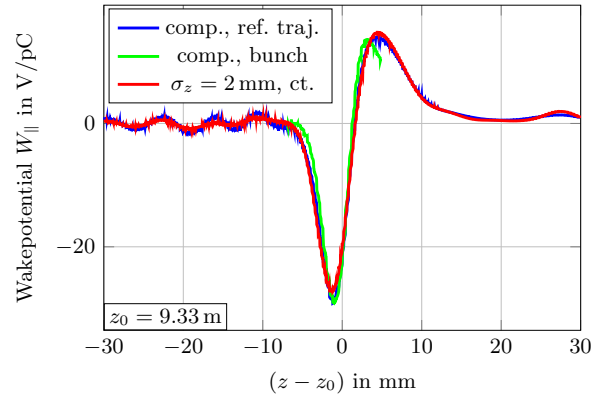


Figure 5: Simulated longitudinal wake potential after traversing BC0 with compression on the reference trajectory (blue) and inside the bunch (green), and without compression (red).

CONCLUSION

We have developed a new approach to compute indirect CSR wakefields which is capable of handling arbitrary bunch movement and chamber geometry without resorting additional approximations. The presented continuous-charge approach provides an effective way of smoothing the collective far field of the particle bunch. First results for the BC0 of the European XFEL illustrate the impact of the dispersive paths on the created wakefield in contrast to the assumption of a mono-energetic bunch, highlighting the importance of accounting for indirect CSR effects generated in complex beam chambers.

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