HIGHER-ORDER ANALYSIS FOR THE WAKEFIELDS OF A SINGLE-PLATE CORRUGATED STRUCTURE*

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Abstract

Single-plate corrugated structures are widely used in x-ray free-electron laser facilities. Accurately modeling the corrugated structure wakefields is of prime importance for correctly simulating their dynamics. In this contribution, we propose a third-order Taylor expansion method for analysing and simulating the single-plate corrugated structures. Our findings indicate that for a small distance and a large transverse beam extension, expansion of the longitudinal wake up to the third order is necessary.

INTRODUCTION

X-ray free-electron lasers (XFELs) offer widely tunable x-ray pulses with unprecedented brightness, driving cutting-edge experiments in a broad spectrum of sciences. Tailoring the temporal and spectral properties of the XFEL pulses often requires manipulation of the electron beam phase space through multiple means. Corrugated wakefield structures [1] are a particular type of passive device used at XFELs for beam manipulation [2–7] as well as diagnostics [8–10]. Such devices utilize the interaction of the electron beam with the surrounding metallic corrugations to passively alter the electron beam phase space. Among them, single-plate corrugated structures [11] are usually used to induce time-dependent slice offset of the electron phase space, aiming for short pulse generation or longitudinal phase space diagnostics.

The modeling of the beam dynamics in the wakefield structures relies on the calculation of the point charge wake functions of the structure. For the single-plate structure, analytical forms of the point charge wake functions were given up to the zeroth-order and first-order approximations [11, 12]. In most of the beam dynamics codes like ELEGANT, assumptions are often made as follows: the longitudinal component is transversely uniform, the transverse component can be decomposed into a uniform term plus one dipole/quadrupole term that scales linearly with the position of the drive/test particle.

When the transverse extension of the electron beam is small and the beam distribution is symmetric, the above assumptions are rather accurate. While in some special cases, higher-order analysis is needed. For example, when the energy spread increase induced by the structure is of interest, the dependency of the longitudinal wake function on transverse coordinates should be included. Another example is when there is a large transverse beam tilt, the linear assumption of transverse wake functions should be examined.

In this paper, we present a general higher-order analysis method for the wakefields and apply it to a single-plate corrugated structure. The method is based on the Taylor expansion of the exact longitudinal wake function [13], which can be obtained analytically or numerically. Our method extends the expansion of Ref. [13] to the third order. Transverse components are connected via the Panofsky-Wenzel Theorem [14]. Taking the exact analytical form of wake functions for the single-plate structure obtained in Ref. [12], we were able to identify the contribution of higher-order terms. The results indicate higher order terms should be properly accounted for in beam dynamics tracking.

This paper is organized as follows: we first introduce the Taylor expansion method, then we calculate all expansion coefficients for the single-plate structure, followed by the analysis of higher-order contributions. Finally, we briefly discuss our findings and conclude this paper.

EXPANSION OF POINT CHARGE WAKE

Consider the two-particle model in the Cartesian coordinate system, where the wakefields are excited by the drive particle located at (x_0, y_0) and probed by the test particle located at (x, y). We expand the longitudinal wake function $w_z(x_0, y_0, x, y, s)$ is known around the beam position $(\bar{x}, \bar{y})$ up to the third-order as

$$w_z(x_0, y_0, x, y, s) \approx h_{00} + \sum_{i=1}^{4} h_{0i}\,\Delta_i + \sum_{i=1}^{4}\sum_{j=1}^{4} h_{ij}\,\Delta_i\Delta_j + \sum_{i=1}^{4}\sum_{j=1}^{4}\sum_{k=1}^{4} h_{ijk}\,\Delta_i\Delta_j\Delta_k, \quad (1)$$

where $\Delta_1 = \Delta x_0 = x_0 - \bar{x}$, $\Delta_2 = \Delta y_0 = y_0 - \bar{y}$, $\Delta_3 = \Delta x = x - \bar{x}$, $\Delta_4 = \Delta y = y - \bar{y}$, the coefficients are given by

$$h_{00}(s) = w_z(\bar{x}, \bar{y}, \bar{x}, \bar{y}, s), \quad (2)$$

$$h_{0i}(s) = \frac{\partial w_z}{\partial r_i}(\bar{x}, \bar{y}, \bar{x}, \bar{y}, s), \quad (3)$$

$$h_{ij}(s) = \frac{1}{2}\frac{\partial^2 w_z}{\partial r_i \partial r_j}(\bar{x}, \bar{y}, \bar{x}, \bar{y}, s), \quad (4)$$

$$h_{ijk}(s) = \frac{1}{6}\frac{\partial^3 w_z}{\partial r_i \partial r_j \partial r_k}(\bar{x}, \bar{y}, \bar{x}, \bar{y}, s), \quad (5)$$

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where $\vec{r} = (x_0, y_0, x, y)$ and $i, j, k = 1, 2, 3, 4$. Note that due to the symmetry of partial derivatives, only part of the above coefficients are needed in actual calculations. Besides, since w_z is a harmonic function of transverse coordinates,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) w_z = 0, \qquad (6)$$

we have $h_{22} = -h_{11}$, $h_{44} = -h_{33}$.

The transverse wake functions are related to the longitudinal wake function through the Panofsky-Wenzel Theorem [14] by

$$\frac{\partial}{\partial s} w_x(x_0, y_0, x, y, s) = \frac{\partial}{\partial x} w_z(x_0, y_0, x, y, s) \qquad (7)$$

$$\frac{\partial}{\partial s} w_y(x_0, y_0, x, y, s) = \frac{\partial}{\partial y} w_z(x_0, y_0, x, y, s) \qquad (8)$$

The transverse wakes can then be found through integration over s. Therefore, the full beam dynamics of the wakefield can be modeled by providing the Taylor expansion coefficients, either from analytical or numerical calculations. It is worthwhile to note that h_{00} is the usually used longitudinal wake in tracking codes, and $\int_s h_{04}$, $\int_s 2h_{24}$, and -$\int_s 2h_{33}$ gives the usually called monopole, dipole and quadrupole term in y direction.

SINGLE-PLATE CORRUGATED STRUCTURE

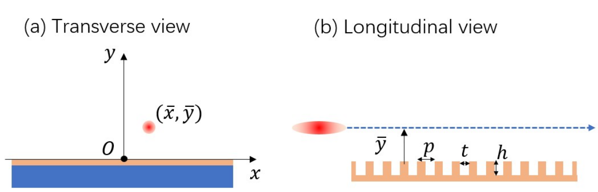

Figure 1: Sketches of a single-plate corrugated wakefield structure, with its transverse view shown in (a) and longitudinal view shown in (b). Corrugations are drawn as a thin layer in the transverse view and shown with corrugation parameters in the longitudinal view. The electron beam is located at $(\bar{x}, \bar{y})$ and shown as a red dot in (a) and a red ellipse in (b).

The single-plate corrugated structure is sketched in Fig.1, with its transverse view shown in Fig.1(a) and longitudinal view shown in Fig.1(b). Corrugation parameters are shown in the longitudinal view. The electron beam is located at $(\bar{x}, \bar{y})$ and shown in red. The electron beam distance toward the corrugations is on the order of a few hundred micrometers, while the electron beam transverse extension could occupy tens to hundreds of micrometers. Using the method of conformal mapping [15, 16], the exact form of w_z at $s = 0^+$ (wake at the origin, also called zeroth-order approximation) was obtained as [12, 17]

$$w_z(x_0, y_0, x, y, s) = -\frac{Z_0 c}{\pi} \Re\left[\frac{1}{(x_0 - x + i(y_0 + y))^2}\right], \qquad (9)$$

while the transverse terms at $s = 0^+$ are given by

$$w_\perp(x_0, y_0, x, y, s) = -\frac{2Z_0 c}{\pi} \frac{s}{(x_0 - x + i(y_0 + y))^3}, \qquad (10)$$

where $w_\perp = w_x + iw_y$ is introduced for simplicity. Here Z_0 is the vacuum impedance, and c is the speed of light in vacuum. In such cases, w_z and the slope $w'_\perp = \partial w_\perp / \partial s$ have no s dependence.

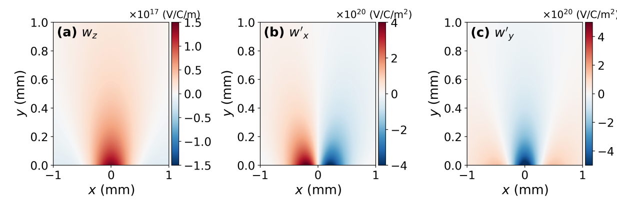

Figure 2: Distribution of the longitudinal wake function and slope of transverse wake functions at $s = 0^+$ for the single-plate corrugated structure.

Assuming that the drive particle is located at $(x_0, y_0) = (\bar{x}, \bar{y}) = (0, 0.5)$ mm, the distribution of longitudinal wake function and the slope of the transverse wake functions are shown in Fig.2. It is clear that for the longitudinal wake, particles at positive Δy and negative Δy experience different longitudinal kick. For w'_x, a focusing effect in x is expected. For w'_y, the pattern can be decomposed as a uniform kick toward the plate and a defocusing effect.

Following Eqs.(2)-(5), we obtain the third-order Taylor expansion coefficients as

$$h_{00} = \frac{cZ_0}{4\pi \bar{y}^2}, \qquad (11)$$

$$h_{02} = h_{04} = -\frac{cZ_0}{4\pi \bar{y}^3}, \qquad (12)$$

$$h_{11} = h_{33} = -\frac{3cZ_0}{16\pi \bar{y}^4}, \qquad (13)$$

$$h_{13} = h_{22} = h_{24} = h_{44} = \frac{3cZ_0}{16\pi \bar{y}^4}, \qquad (14)$$

$$h_{112} = h_{114} = h_{233} = h_{334} = \frac{cZ_0}{8\pi \bar{y}^5}, \qquad (15)$$

$$h_{123} = h_{134} = h_{222} = h_{224} = h_{244} = h_{444} = -\frac{cZ_0}{8\pi \bar{y}^5} \qquad (16)$$

Here we have omitted terms that satisfy the exchange of derivative order, like h_{321} and terms that are zero.

HIGHER-ORDER CONTRIBUTION

Using the expansion in Eq.(1), we collected the contribution of each expansion term up to a certain order, shown as a function of position deviation from the beam for the drive and test particles. The results are listed in Table.1. A few general observations can be obtained. First, the third-order expansion method gives the second-order approximation for transverse wakes. This is straightforward from the Panofsky-Wenzel Theorem. Second, for a specific beam transverse distribution, some higher-order terms could be cancelled if

Table 1: Contributions to w_z, w'_x, and w'_y by Expansion Order

	w_z	w'_x	w'_y
0th	$\frac{cZ_0}{4\pi\bar{y}^2}$	0	0
1st	$-\frac{cZ_0(\Delta y+\Delta y_0)}{4\pi\bar{y}^3}$	0	$-\frac{cZ_0}{4\pi\bar{y}^3}$
2nd	$-\frac{3cZ_0\left[(\Delta x-\Delta x_0)^2-(\Delta y+\Delta y_0)^2\right]}{16\pi\bar{y}^4}$	$\frac{3cZ_0(-\Delta x+\Delta x_0)}{8\pi\bar{y}^4}$	$\frac{3cZ_0(\Delta y+\Delta y_0)}{8\pi\bar{y}^4}$
3rd	$-\frac{cZ_0(\Delta y+\Delta y_0)\left[-3(\Delta x-\Delta x_0)^2+(\Delta y+\Delta y_0)^2\right]}{8\pi\bar{y}^5}$	$\frac{3cZ_0\,(\Delta x-\Delta x_0)(\Delta y+\Delta y_0)}{4\pi\bar{y}^5}$	$\frac{3cZ_0\left[(\Delta x-\Delta x_0)^2-(\Delta y+\Delta y_0)^2\right]}{8\pi\bar{y}^5}$

$\Delta y+\Delta y_0$ or $\Delta x-\Delta x_0$ can be made to be zero for all drive-test particle pairs. This is usually possible if the beam has a symmetric distribution.

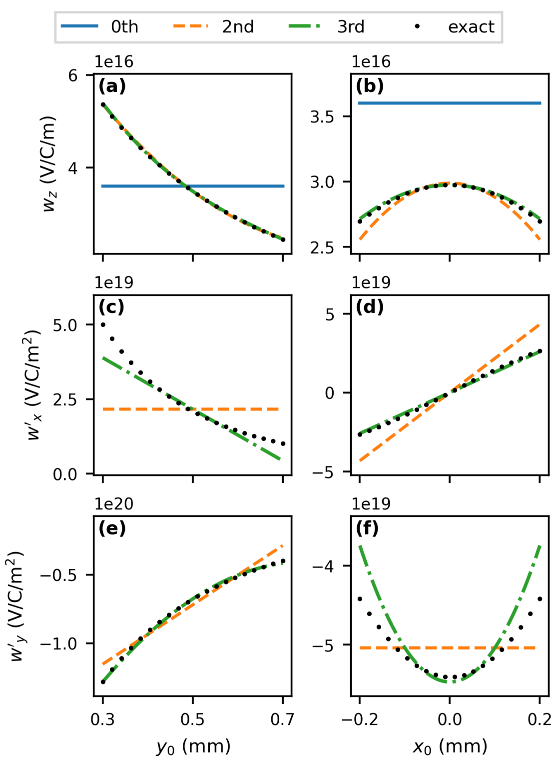

Figure 3: w_z, w'_x, and w'_y as a function of drive particle positions, with calculations of different approximation orders (lines) and comparison to the results from the exact form (dots). (a), (c), (e) are calculated for $\Delta x=\Delta y=0$ mm, $\Delta x_0=0.1$ mm. (b), (d), (f) are calculated for $\Delta x=\Delta y=0$ mm, $\Delta y_0=0.1$ mm.

For a simpler investigation, we set $\Delta x=\Delta y=0$, and perform a scan of drive particle position in Fig.3, with the results of Δy_0 scan (Δx_0 fixed to 0.1 mm) in Fig.3(a,c,e) and the results of Δx_0 scan (Δy_0 fixed to 0.1 mm) in Fig.3(b,d,f). It can be seen that for w_z, w'_x, and w'_y, the expansion up to the third order gives more accurate wake functions compared with the usually used lower order approximations. This indicates that for cases with a large beam size and asymmetric distribution, higher-order terms should be included in the beam tracking.

Figure 4 shows the observed relative error compared with the exact form for different electron beam positions $\bar{y}$. The errors are observed at $\Delta x=\Delta y=0$ mm for (a) fixed Δy_0 and (b) fixed Δx_0. The results indicate that the errors could be up to a few tens of percent for a small distance to the plates and a large beam extension.

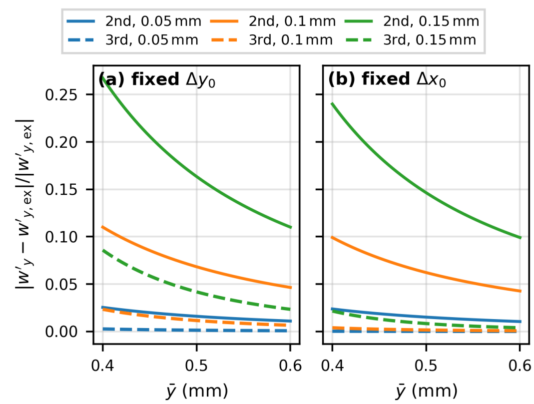

Figure 4: Relative error observed at $\Delta x=\Delta y=0$ mm while changing beam position, for the case of (a) fixed $\Delta y_0=0.05, 0.1, 0.15$ mm, $\Delta x_0=0$ mm, (b) fixed $\Delta x_0=0.05, 0.1, 0.15$ mm, $\Delta y_0=0$ mm.

CONCLUSION

In this paper, we have examined the higher-order behavior of the point charge wake functions for a single-plate corrugated structure using the Taylor expansion method. Our results show that expansion up to the third order should be considered when the beam is close to the plates and has a large transverse extension, where errors could be as high as tens of percent. The findings could largely benefit the accurate modeling of beam dynamics in corrugated structure applications, such as short pulse generation and longitudinal phase space diagnostics. Although we only considered the single-plate structures with zeroth-order wake, the analysis could be easily extended to other transverse shapes as well as the first-order wake approximation [12]. The established analysis method is general and could be further applied to the analysis of other wakefield-related situations, such as in low-energy photo-injectors, where the beam sizes tend to be large. Further study could be the implementation of these higher-order terms into tracking codes like OCELOT [18] and investigating the higher-order effect on the whole wake.

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