SUPERPOSITION OF MULTIPOLAR TRANSVERSE ELECTRIC MODES IN A UNIFIED RF STRUCTURE

O. Betteridge*,[1,3], R. J. Apsimon[2,3], and Ö. Apsimon[1,3]
[1]Department of Physics and Astronomy, The University of Manchester, Manchester, United Kingdom
[2]Department of Engineering, Lancaster University, Lancaster, United Kingdom
[3]Cockcroft Institute of Accelerator Science and Technology, Daresbury, United Kingdom

Abstract

A field-driven method is presented for synthesising multipolar transverse-electric cavities. Cylindrical standing-wave TE solutions are superimposed analytically, and the cavity surface is reconstructed by enforcing perfect-electric-conductor boundary conditions. The method produces closed hybrid cavity geometries whose modal content is validated against CST eigenmode simulations. Two examples are shown: the superposition of the $TE_{01} + TE_{21}$ and $TE_{21} + TE_{41}$ hybrid cavities. In both cases, the intended azimuthal field structure is preserved with sub-percent-level agreement between the analytic model and CST. The method provides a route to structured TE-mode RF cavities for future beam-manipulation studies.

INTRODUCTION

Structured RF cavities provide additional degrees of freedom for beam manipulation beyond those available from conventional axisymmetric accelerating modes. Multipolar TM-mode cavities have recently been developed using azimuthal modulation methods [1]. An equivalent construction for transverse-electric, or TE, modes is useful because TE fields can provide structured transverse magnetic and electric fields while retaining no longitudinal electric field in the ideal basis. TE-like modes are already used in accelerator structures such as H-mode drift-tube linacs, where magnetic-field-dominated resonators provide efficient acceleration at low particle velocity [2–6]. TE modes have also been used in studies of RF breakdown, where surface electric and magnetic fields can be separated more cleanly than in conventional accelerating modes [7]. In the beam-dynamics context, RF magnetic-mode concepts have also been proposed for emittance compensation in superconducting RF guns [8]. These examples motivate a systematic method for designing TE cavities with prescribed multipolar content. This paper presents a surface-synthesis method for hybrid TE cavities; a more detailed treatment is given in Ref. [9]. Analytical TE standing-wave solutions are superimposed in cylindrical coordinates, and the cavity boundary is found by enforcing the perfect-electric-conductor condition on the resulting field.

FIELD MODEL

The TE basis is constructed from cylindrical standing-wave solutions [10–13]. The longitudinal magnetic field is written as

$$H_z(r,\phi,z) = A_m J_m(\gamma r)\sin(m\phi+\phi_0)\sin\left[\beta(z-L/2)\right],$$

where J_m is a Bessel function of the first kind, m is the azimuthal index, γ is the transverse propagation constant, and β is the longitudinal propagation constant. The wave numbers satisfy

$$k^2 = \gamma^2 + \beta^2,$$

with $k = \omega/c$.

For a circular TE cavity, the perfect-electric-conductor boundary gives the usual eigenfrequency condition

$$\left(\frac{x'_{mn}}{a}\right)^2 + \left(\frac{p\pi}{L}\right)^2 = \left(\frac{\omega}{c}\right)^2,$$

where x'_{mn} is the nth zero of J'_m, a is the cavity radius, and p is the longitudinal index. In this work $p = 1$, and the notation $TE_{\{m_1,m_2\}1}$ denotes a hybrid formed from TE solutions with azimuthal indices m_1 and m_2.

A direct superposition of different circular-cavity TE eigenmodes is not self-consistent, since each mode generally requires a different radius for a common frequency. Instead, the fields are first treated as an analytic basis in dimensionless coordinates, and the physical boundary is reconstructed afterwards by enforcing the conductor condition.

SURFACE SYNTHESIS

The unknown cavity cross-section is represented by a truncated Fourier series,

$$r(\phi) = a_0 + \sum_{\nu=1}^{\nu_{\max}}\left[a_\nu\cos(\nu\phi) + b_\nu\sin(\nu\phi)\right].$$

This representation naturally enforces smoothness and periodicity of the closed boundary. The design variables are the Fourier coefficients, together with possible transverse offsets.

For a non-circular TE cavity, the boundary cannot be found by imposing $E_\phi = 0$ alone. Both radial and azimuthal electric-field components contribute to the tangential field at the surface. The surface is therefore required to satisfy

$$\mathbf{E}\cdot\mathbf{S} = 0$$

at all sampled azimuthal locations, where $\mathbf{S}$ is the tangent vector to the boundary.

The problem is solved as a feasibility optimisation using nonlinear constrained optimisation methods [14]. In other

* oliver.betteridge@postgrad.manchester.ac.uk

words, we enforce the tangential electric field to vanish on the boundary, i.e., $\mathbf{E} \cdot \mathbf{S} = 0$, where $\mathbf{E}$ is the electric field sampled at discrete angles on the boundary and $\mathbf{S}$ is the tangent vector of the boundary. This corresponds to the perfect electric conductor (PEC) boundary condition. A smooth weighting is applied near global field zeros to improve numerical robustness, since these locations otherwise provide poor gradient information to the optimiser.

$TE_{\{0,2\}1}$ HYBRID CAVITY

As a first example, TE_{01} and TE_{21} solutions are combined with equal weights and zero relative phase. The resulting boundary follows a smooth, elliptical-like contour of the hybrid field and satisfies the conductor condition to numerical precision. The final geometry is shown in Fig. 1.

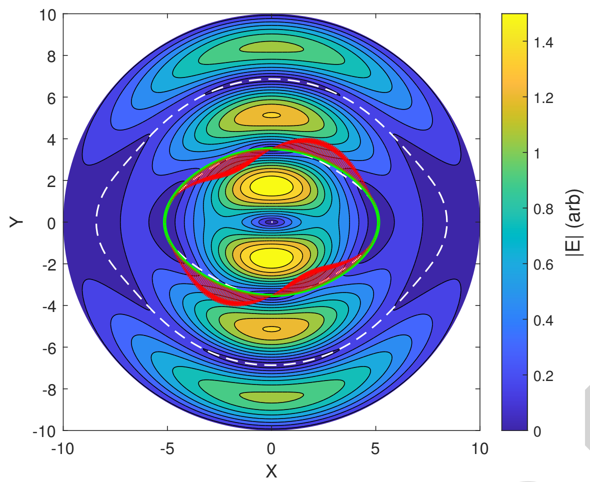

Figure 1: Synthesised $TE_{\{0,2\}1}$ hybrid cavity cross-section. The electric-field magnitude is shown with field quivers, dashed contours indicate zeros of E_ϕ, and the solid boundary is the reconstructed cavity surface.

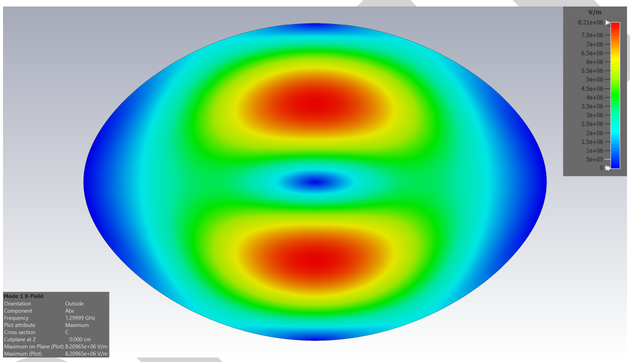

Figure 2: Synthesised $TE_{\{0,2\}1}$ hybrid cavity in CST.

The boundary determined through this procedure is then imported into CST [15] and rescaled to a target frequency. To correctly establish the length and frequency of the cross-sectional boundary, we first assign a non-zero length to form a complete 3D structure in CST and then switch the boundary condition applied to the end caps from PEC to PMC (perfect magnetic conductor) conditions. Running an eigenmode simulation in CST determines the transverse cut-off frequency, from which the physical cavity length is obtained for the desired longitudinal index and frequency. However, there are cases in which the cavity cannot be realised at a certain frequency because it falls below the cut-off frequency; in such cases, the cross-sectional boundary must also be rescaled so that the new cut-off frequency lies below the desired frequency. Details of this scaling procedure, from a 2D cross-section to a fully 3D realisable cavity, are given in Ref. [16]. The CST mode is shown in Fig. 2.

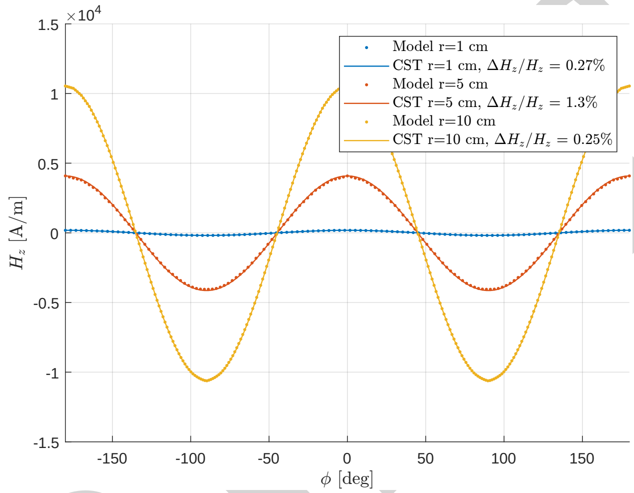

Figure 3: Comparison of H_z between CST and the analytic model for the $TE_{\{0,2\}1}$ hybrid mode at different radii.

To validate the modal content, the longitudinal magnetic field H_z from CST is compared with the analytic model on circular contours in the cavity mid-plane. Figure 3 shows agreement at the sub-percent level, confirming that the intended TE_{01}–TE_{21} superposition is preserved in the CST eigenmode.

$TE_{\{2,4\}1}$ HYBRID CAVITY

The second example combines TE_{21} and TE_{41} solutions with equal weights and a relative phase of $\pi/4$. The phase is chosen to align the radial zero-crossings of each mode. More specifically, the TE_{21} fields have radial zeros that resemble a plus sign, whereas the TE_{41} fields have two additional zero-crossings that pass through the diagonals, forming a star-like shape. The rotation ensures that the TE_{21} fields overlap with the TE_{41} fields, producing only four unique radial nodes instead of six. This case is more challenging because the field contains multiple neighbouring zero-contour branches. The optimiser can therefore encounter regions where the desired closed contour approaches another solution branch or a global field zero. These features make the TE synthesis problem more numerically delicate than the corresponding TM case [1].

Despite this difficulty, a smooth two-fold-symmetric cavity boundary is obtained. Figure 4 shows the corresponding MATLAB field contour and reconstructed cavity surface.

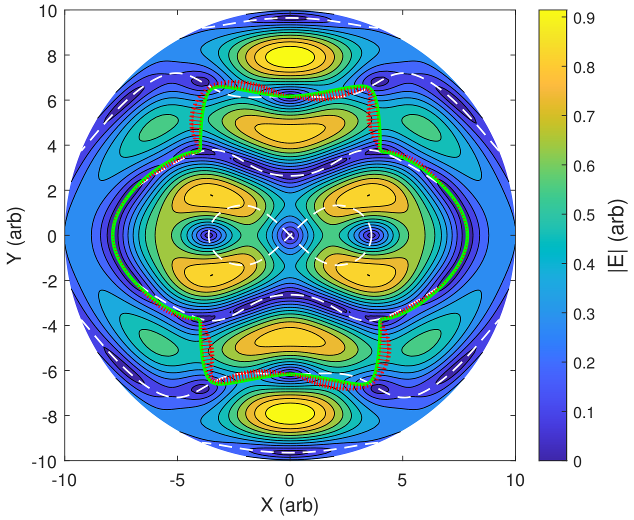

Figure 4: Synthesised $TE_{\{2,4\}1}$ hybrid cavity cross-section from the MATLAB surface-synthesis model. The electric-field magnitude is shown with field quivers, dashed contours indicate zeros of E_{ϕ}, and the solid boundary is the reconstructed cavity surface.

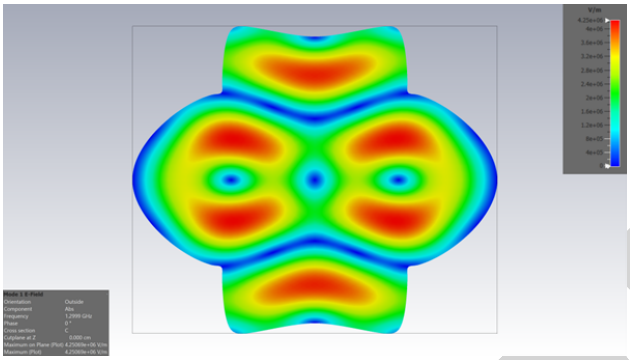

Figure 5: Synthesised $TE_{\{2,4\}1}$ hybrid cavity in CST.

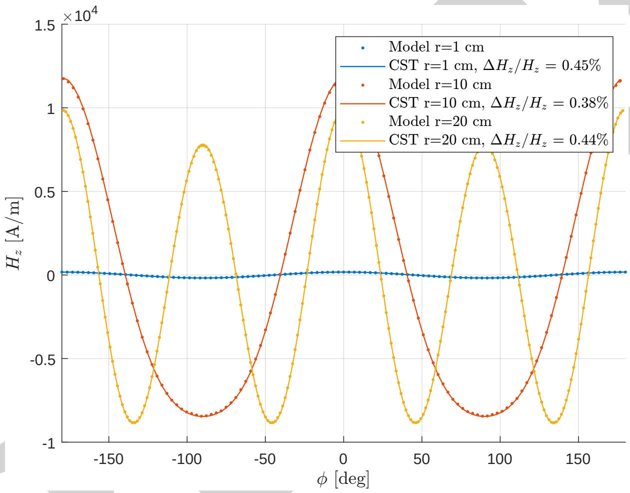

Figure 6: Comparison of H_z between CST and the analytic model for the $TE_{\{2,4\}1}$ hybrid mode at different radii.

The same CST rescaling procedure gives an operating frequency close to 1.3 GHz. c 5 shows the same mode constructed in MATLAB and reproduced in CST, while the modal-purity analysis in Fig. 6 shows that no higher-order modal content is introduced and that the intended TE_{21}-TE_{41} structure is retained after surface synthesis. This example demonstrates that the method can generate higher-order hybrid TE geometries.

CONCLUSION

A field-driven method has been demonstrated for synthesising multipolar TE-mode cavities. The approach uses analytical cylindrical TE solutions as a basis, superimposes selected multipolar components, and reconstructs a closed cavity boundary by enforcing the PEC boundary condition. The method has been validated for $TE_{\{0,2\}1}$ and $TE_{\{2,4\}1}$ hybrid cavities. In both cases, CST eigenmode simulations preserve the intended modal structure, with sub-percent-level agreement in the longitudinal magnetic field when compared with the analytic model. The $TE_{\{2,4\}1}$ example highlights the increased numerical difficulty of TE surface synthesis, especially near global field zeros and neighbouring solution branches. This work establishes a practical route to multipolar TE cavity design. Future studies will extend the method to beam pipes and to beam dynamics applications such as emittance correction and chromatic compensation [17].

ACKNOWLEDGEMENTS

The authors acknowledge the support of the Cockcroft Institute Core Grant, funded by UKRI under grant number UKRI1887.

REFERENCES

[1] L. M. Wroe, S. L. Sheehy, and R. J. Apsimon, "Creating exact multipolar fields with azimuthally modulated RF cavities", *Phys. Rev. Accel. Beams*, vol. 25, no. 6, p. 062001, 2022. doi:10.1103/PhysRevAccelBeams.25.062001

[2] F. Gerigk, "Cavity types", 2011, arXiv: 1111.4897 [physics.acc-ph], https://arxiv.org/abs/1111.4897,

[3] E. Jensen, "Cavity basics", 2012, arXiv: 1201.3202 [physics.acc-ph], https://arxiv.org/abs/1201.3202,

[4] J. Tamura, Y. Kondo, T. Morishita, F. Naito, and M. Otani, "Acceleration efficiency of TE-mode structures for proton linacs", in *Proc. LINAC'22*, 2022. doi:10.18429/JACoW-LINAC2022-MOPOGE13

[5] M. Basten *et al.*, "Continuous wave interdigital H-mode cavities for alternating phase focusing heavy ion acceleration", *Rev. Sci. Instrum.*, vol. 93, p. 063303, 2022. doi:10.1063/5.0094859

[6] I. M. Kapchinskii and V. A. Teplyakov, "A linear ion accelerator with spatially uniform hard focusing", *Prib. Tekh. Eksp.*, vol. 1970, no. 2, pp. 19–22, 1970.

[7] V. A. Dolgashev, J. Neilson, S. G. Tantawi, and A. D. Yeremian, "A dual-mode accelerating cavity to test RF breakdown dependence on RF magnetic fields", in *Proc. IPAC'11*, pp. 247–249, 2011.

[8] K. Flöttmann, D. Janssen, and V. Volkov, "Emittance compensation in a superconducting RF gun with a magnetic mode", *Phys. Rev. ST Accel. Beams*, vol. 7, no. 9, p. 090702, 2004. doi:10.1103/PhysRevSTAB.7.090702

[9] O. Betteridge, R. J. Apsimon, and O. Apsimon, "Surface synthesis of multipolar TE-mode cavities", *Phys. Rev. Accel. Beams*, 2026. doi:10.1103/gjzt-fvmn

[10] J. D. Jackson, *Classical electrodynamics*. New York, NY, USA: John Wiley & Sons, 1999.

[11] S. Ramo, J. R. Whinnery, and T. Van Duzer, *Fields and waves in communication electronics*. New York, NY, USA: John Wiley & Sons, 1994.

[12] D. M. Pozar, *Microwave engineering*. Hoboken, NJ, USA: John Wiley & Sons, 2012.

[13] R. E. Collin, *Field theory of guided waves*. New York, NY, USA: IEEE Press, 1991. doi:10.1109/9780470544648

[14] P. T. Boggs and J. W. Tolle, "Sequential quadratic programming", *Acta Numer.*, vol. 4, pp. 1–51, 1995. doi:10.1017/S0962492900002518

[15] *CST Studio Suite: electromagnetic field simulation software*, Dassault Systèmes, 2023. https://www.3ds.com/products-services/simulia/products/cst-studio-suite

[16] O. Bet, "Multipolar TE modes", https://github.com/OliverBet/Multipolar_TE_modes.git, Data supporting the findings of this article are openly available in this repository, 2026,

[17] L. M. Wroe, W. Wuensch, and R. J. Apsimon, "Controlling the transverse multipole components in RF cavity modes using the azimuthal modulation method", *Phys. Rev. Accel. Beams*, vol. 28, no. 8, p. 082002, 2025. doi:10.1103/tjgp-gjq7