A SPATIAL BORIS-LIKE SCHEME FOR PARTICLE TRACKING IN MAGNETOSTATIC FIELDS: IMPLEMENTATION AND PROPERTIES

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Abstract

Accurate and stable integration of charged-particle motion in complex magnetic fields is essential for beam-dynamics simulations in accelerators and beam lines. For the Xsuite simulation framework we have recently developed a spatially discretized Boris-like algorithm that advances particle coordinates using the longitudinal position as the independent variable. The scheme retains the symmetric kick–rotate–kick structure of the standard time-based Boris pusher, ensuring exact preservation of the total momentum magnitude and of the phase-space volume. We derive its formal properties using operator-splitting and backward-error analysis, showing that it is second-order accurate, and that the symplectic error scales quadratically with the integration step. Such properties are also verified by numerical tests on a representative field distribution.

INTRODUCTION

The Boris algorithm [1] is widely used for charged-particle tracking because it is explicit, second-order accurate, volume preserving, and robust over long integrations [2, 3]. In accelerator and beam-line simulations, however, the natural independent variable is often the longitudinal coordinate rather than time. Spatial Boris variants have therefore been considered for beam applications [4, 5].

We have developed a spatial Boris-like integrator for tracking in arbitrary magnetostatic field distributions within the Xsuite simulation framework [6, 7]. The method advances particles using the longitudinal coordinate z as independent variable, while retaining a symmetric drift–kick–rotate–kick–drift structure. This paper summarizes the algorithm, the main preservation and accuracy properties, and numerical checks performed with a solenoid field including fringe terms; full derivations are given in Ref. [8].

SPATIAL FORM OF THE EQUATIONS

For a particle of charge q and momentum P in a magnetostatic field, we can write:

$$\frac{dP}{dt} = qv \times B, \qquad v = \frac{P}{\gamma m}. \tag{1}$$

The total momentum $P = |P|$ is constant. Assuming that the longitudinal component of the momentum P_z does not vanish, the equations can be rewritten using z as independent variable:

$$\frac{dP}{dz} = \frac{q}{P_z} P \times B, \qquad \frac{dr}{dz} = \frac{P}{P_z}. \tag{2}$$

If the magnitude of the momentum is known, the longitudinal component can be obtained from the transverse components as

$$P_z = \sqrt{P^2 - P_x^2 - P_y^2} > 0. \tag{3}$$

In accelerator variables the elapsed time satisfies

$$\frac{dt}{dz} = \frac{\gamma m}{P_z}. \tag{4}$$

BORIS-LIKE SPATIAL STEP

Let $h = \Delta z$ be the step size. The Boris-like step is built as a symmetric drift–kick–rotate–kick–drift composition, defined by the following steps.

Step 1: first half drift

Starting from (x, y, z, P_x, P_y), compute P_z from Eq. (3) and drift the transverse coordinates by half a longitudinal step:

$$x_h = x + \frac{h}{2}\frac{P_x}{P_z}, \qquad y_h = y + \frac{h}{2}\frac{P_y}{P_z}, \qquad z_h = z + \frac{h}{2}. \tag{5}$$

Step 2: field evaluation

The magnetic field $B = (B_x, B_y, B_z)$ is evaluated at (x_h, y_h, z_h).

Step 3: first half kick

The transverse field components B_x and B_y give the first half kick:

$$P_x^- = P_x - \frac{qh}{2}B_y, \qquad P_y^- = P_y + \frac{qh}{2}B_x, \tag{6}$$

Step 4: midpoint longitudinal momentum

The longitudinal momentum used in the central rotation is reconstructed after the first half kick:

$$P_{z,\mathrm{mid}} = \sqrt{P^2 - (P_x^-)^2 - (P_y^-)^2}. \tag{7}$$

Step 5: Boris rotation

The longitudinal field component B_z generates a rotation in the transverse momentum plane. The exact frozen-field rotation angle is approximated with an $O(h^3)$ error, using the following rational coefficients:

$$T = \frac{qB_z h}{2P_{z,\mathrm{mid}}}, \qquad S = \frac{2T}{1 + T^2}, \qquad C = \frac{1 - T^2}{1 + T^2}. \tag{8}$$

The momenta after the rotation are

$$P_x^+ = CP_x^- + SP_y^-, \tag{9}$$

$$P_y^+ = -SP_x^- + CP_y^-. \tag{10}$$

This is a planar rotation by angle $2\arctan T$, and therefore preserves $(P_x^-)^2 + (P_y^-)^2$ exactly during the B_z substep. Since C and S are computed directly from T, no trigonometric functions need to be evaluated.

Step 6: second half kick

The same transverse kick is applied a second time:

$$P_x' = P_x^+ - \frac{qh}{2}B_y, \qquad P_y' = P_y^+ + \frac{qh}{2}B_x. \tag{11}$$

The final longitudinal momentum is again reconstructed algebraically:

$$P_z' = \sqrt{P^2 - (P_x')^2 - (P_y')^2}. \tag{12}$$

Step 7: second half drift

The step is completed by drifting the coordinates with the final momentum:

$$x' = x_h + \frac{h}{2}\frac{P_x'}{P_z'}, \qquad y' = y_h + \frac{h}{2}\frac{P_y'}{P_z'}. \tag{13}$$

The final longitudinal coordinate is $z' = z + h$.

Step 8: time update

The arrival-time update is performed as a midpoint approximation using the longitudinal momentum after the first half kick:

$$\Delta t = \frac{\gamma m h}{P_{z,\text{mid}}}, \tag{14}$$

so that

$$t' = t + \Delta t. \tag{15}$$

PROPERTIES OF THE ALGORITHM

Several properties of the spatial Boris-like map can be established mathematically. In this section we report the main results relevant for tracking applications; the full derivations and assumptions are given in the extended note [8].

Momentum Preservation

The method enforces the magnetic-only invariant $P = |P|$ at every substep. The central Boris update is a rotation in the (P_x, P_y) plane and therefore preserves $(P_x^-)^2 + (P_y^-)^2$ during the B_z part of the step. The transverse kicks can change the split between transverse and longitudinal momentum, but P_z is not advanced by a separate finite-difference equation. It is reconstructed algebraically from Eq. (3) after the first half kick and again after the second half kick. Thus, up to roundoff, the numerical state always satisfies

$$(P_x)^2 + (P_y)^2 + (P_z)^2 = P^2. \tag{16}$$

This is the spatial analogue of the exact kinetic-energy preservation of the standard time-based Boris pusher in a pure magnetic field.

Phase-Space Volume Preservation

Let Φ_h denote one step of the algorithm with step size h, restricted here to the reduced variables (x, y, P_x, P_y), and let $D\Phi_h$ be its Jacobian with respect to these variables. The map Φ_h is exactly volume preserving. Each elementary substep has unit determinant: the drifts are block triangular maps with identity blocks, the kicks are shears in momentum, and the central rotation preserves the area element $dP_x\,dP_y$. Therefore the one-step Jacobian satisfies

$$\det D\Phi_h = 1. \tag{17}$$

The same argument extends to (x, y, P_x, P_y, t, E). Since E is constant and t is advanced by a shear, the extended Jacobian remains block triangular and has determinant one.

Reversibility

The step is reversible because applying the same algorithm with step $-h$ first reverses the second half drift, evaluates the field at the same midpoint, undoes the second half kick, applies the inverse rotation, undoes the first half kick, and finally reverses the first half drift. Hence

$$\Phi_{-h} = \Phi_h^{-1}. \tag{18}$$

Accuracy

For smooth magnetic fields, and as long as P_z stays positive, all operations in the one-step map are smooth functions of the initial condition and of h. Expanding the update for small h shows that its leading term is the spatial Lorentz equation, Eq. (2). As a consequence of the reversibility, the even term in the local defect cancels, so the one-step error is $O(h^3)$. For a fixed length $L = Nh$, where N is the number of steps, the global map and its Jacobian are therefore second-order accurate:

$$\Phi_h^L - \varphi_L = O(h^2), \qquad D\Phi_h^L - D\varphi_L = O(h^2). \tag{19}$$

The asymptotic notation used here is defined as follows: $R_h = O(h^p)$ means that the norm of the remainder R_h is bounded by Ch^p for $0 < h < h_{\max}$, with constants C and $h_{\max}$ independent of the initial condition in the region considered.

Boundary Symplectic Defect

The reduced mechanical variables are not canonical inside a magnetic field. Nevertheless, if the entrance and exit are field-free, canonical and mechanical transverse momenta coincide at both boundaries. The exact entrance-to-exit transfer map is then symplectic in the boundary variables $(x, p_x, y, p_y, t, -E)$. We define the boundary symplectic defect of the numerical map as the matrix residual $M^T SM - S$, where M is the Jacobian of the numerical full-length map and S is the canonical symplectic matrix. Inserting Eq. (19) into this residual gives

$$M^T SM - S = O(h^2), \tag{20}$$

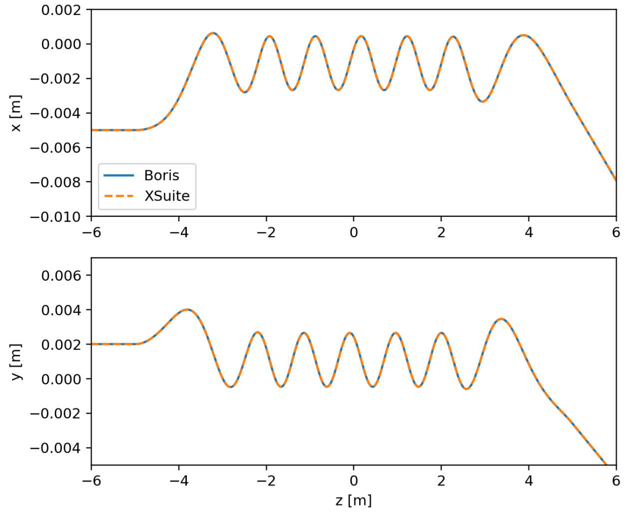

Figure 1: Transverse position from the spatial Boris-like integrator and from the Xsuite variable-solenoid map.

NUMERICAL CHECKS

The method was tested with a linear-fringe solenoid, for which an independent reference map is available in Xsuite. This example is simple enough to isolate the convergence properties, but still exercises all parts of the algorithm: the body field produces the longitudinal-field rotation, while the entrance and exit fringes produce transverse kicks. The longitudinal field used for the test is

$$B_z(s) = \begin{cases} B_0(s-s_1)/(s_2-s_1), & s_1 \le s < s_2, \\ B_0, & s_2 \le s < s_3, \\ B_0(s_4-s)/(s_4-s_3), & s_3 \le s < s_4, \\ 0, & \text{otherwise,} \end{cases} \tag{21}$$

with $B_0 = 1$ T and $(s_1, s_2, s_3, s_4) = (-5, -2, 2, 5)$ m. The corresponding transverse fields are

$$B_x = -\frac{x}{2}\frac{dB_z}{ds}, \qquad B_y = -\frac{y}{2}\frac{dB_z}{ds}. \tag{22}$$

The field vanishes outside the interval $[s_1, s_4]$, so the entrance and exit boundaries are field free and the boundary symplectic defect of Eq. (20) is well defined. An electron with reference energy 50 MeV was tracked from $x_0 = -10^{-3}$ m, $y_0 = 10^{-3}$ m, $p_{x0} = -10^{-3}$, and $\delta_0 = 0$. Figure 1 compares the trajectory with the existing Xsuite solenoid map.

The number of steps was varied from 200 to 100000. The exit-position error was computed with respect to the finest-step result, and the boundary symplectic deviation was measured as $\|M^T SM - S\|_2$ from the full entrance-to-exit response matrix. These two diagnostics probe different aspects of the method: the trajectory error checks the second-order accuracy of the map, while the symplectic deviation checks the quadratic scaling of the boundary symplectic defect. Both observables show the expected quadratic scaling with h, as shown in Figs. 2 and 3.

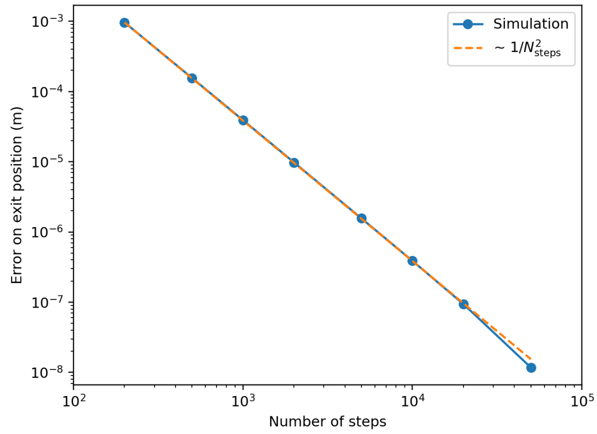

Figure 2: Convergence of the exit-position error. The dashed line shows a $1/N^2_{\text{steps}}$ reference scaling.

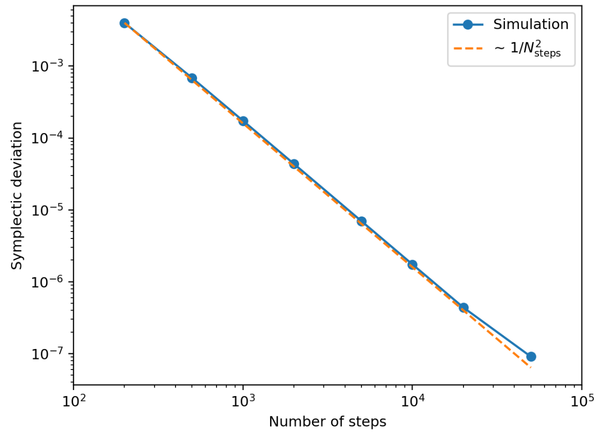

Figure 3: Convergence of the boundary symplectic deviation $\|M^T SM - S\|_2$. The dashed line shows a $1/N^2_{\text{steps}}$ reference scaling.

CONCLUSION

A spatial Boris-like scheme has been implemented for particle tracking in general static magnetic fields in Xsuite. The algorithm uses z as independent variable while retaining a symmetric Boris structure. Its main properties have been shown mathematically: it preserves the total momentum magnitude by construction, is exactly volume preserving in the (x, y, P_x, P_y) and (x, y, P_x, P_y, t, E) phase spaces, and is reversible and second-order accurate for smooth fields. For field-free entrance and exit boundaries, the symplectic defect of the full transfer map scales as $O(h^2)$. Numerical tests with a solenoid including fringe fields confirm the expected convergence of both the trajectory error and the symplectic deviation. The implementation has already been used to model insertion devices, including undulator magnets for synchrotron light sources and colliders; this application is described in a separate contribution to IPAC'26 [9].

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