STATISTICAL PROPERTIES OF ATTOSECOND SASE FELs

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Abstract

Free electron lasers can now generate xray pulses with durations in the attosecond regime. Optimal utilization of the short pulses for attosecond science necessitates precise measurement of the pulse durations which there are promising methods to achieve. However, these are developing experimental techniques, not yet routine procedure. Until they mature, there is a wealth of information in the measured pulse energies and xray spectra, which are easily obtained. We show that while this information is insufficient to draw any conclusions about a single shot, with a large enough dataset, the statistics can determine length of the electron bunches, on average. FEL theory then predicts the distribution of the xray pulse durations. To achieve this, we extend the classical theory put in place by Saldin and Bonifacio to include short bunches with arbitrary current profiles, deriving integral expressions for some key statistical observables. The analytical is approach compared to 1D nonlinear simulations, showing good agreement until saturation sets in.

INTRODUCTION

The behavior of most of the matter we interact with on earth depends strongly on the distributions of bound electrons and the correlations between them [1–3]. These electrons move and redistribute on time scales of femtoseconds and attoseconds. Being able to access these scales directly with radiation opens up completely new avenues of physics [4]. One important tool for this purpose is the free electron laser (FEL) [5], an instrument that can produce attosecond xray pulses with extreme peak power. [6–11] At this moment, the only way to achieve attosecond hard x-ray pulses at an FEL is in the SASE [12, 13] mode (Self Amplified Spontaneous Emission) which works by exponentially amplifying incoherent undulator radiation. [14] SASE is an inherently random process and thus the properties of the xray pulses vary significantly between shots. In particular, the duration of the pulses can fluctuate dramatically. [15] In addition to the natural randomness from SASE, the duration of the electron bunches themselves also fluctuates due to jitter in the LINAC and the strong bunch compression required. [16] To use the FEL to its fullest potential, ultra-fast experiments ideally require shot by shot direct measurement of the pulse durations. The leading method is currently angular streaking, which has been successful in characterizing attosecond UV sources based on high-harmonic generation and has been demonstrated for FELs. [3, 17, 18] For attosecond xray FELs it is still practically complicated and requires significant computation. Direct measurements of the electron bunch durations is also non-trivial because the resolution is usually limited by RF frequency in transverse deflecting cavities. [19,20] What is readily available at most xray FEL facilities is the xray spectrum and pulse energy. The statistics of those measurements carry information about the electron bunches and there are succesful demonstrations of reconstructing relatively long bunch lengths at FLASH and LCLS. As seen in Refs. [15, 21, 22], one can for example use the variance of the pulse energy or spectral intensity correlations. Bunch lengths have also been diagnosed with the spectrum of spontaneous undulator radiation. [23] In a heuristic sense, xray spectra are also commonly used when tuning FEls – seeing wide, smooth spectra indicates that the (lasing part of) electron bunches are sufficiently short. For long uniform bunches, there is a complete analytical description of the statistics [24, 25], but for short bunches only numerical results are available. [15] Our contribution is generalizing the analytical picture, providing a wide description of the statistics of short bunches with an arbitrary current profile. Based on analytical calculations and numerical data, we draw these conclusions: 1) The average bunch length can be estimated from the mean and variance of the mean bandwidth. 2) While the estimate from energy statistics is most sensitive at long bunch lengths, the estimate from bandwidths is sensitive at short bunch lengths. 3) There exists a specific bunch length such that the pulse duration is very stable. 4) A single spectrum gives no information about the pulse duration or bunch length, even when there is no chirp in the bunch.

Analytical Model and Statistics

As a basis for analytical development we use the linearized, dimensionless 1D collective variables equations from Ref. [26]:

$$\begin{cases} \partial_\zeta^2 b = ia, \\ (\partial_\zeta + \partial_\tau)a = q(\tau)b, \\ b(\zeta = 0, \tau) = b_0(\tau), \end{cases} \tag{1}$$

with the remaining boundary conditions being homogeneous at $\zeta = 0$ and $\tau = 0$. Here a, b are the field envelope and bunching factor and ζ, τ are the distances along the undulator and along the bunch, scaled by the gain length $l_g = \lambda_u/(4\pi\varrho)$ and cooperation time $\ell_c = \lambda_r/(4\pi\varrho)$ respectively. ϱ is the usual Pierce parameter [26] with 3D corrections included and λ_u, λ_r are the spatial periods of the undulator and the radiation. q is the current profile, normalized to a peak value of 1 and we take it to be an arbitrary continuous function with support on $(0, \tau_b)$, so that τ_b is dimensionless full-length of the bunch. Since we are interested in SASE, the initial bunching factor b_0 is a random variable modeling shot noise. The dimensionless power is related to real units through the Pierce parameter: $P_{\text{real}} = \varrho|a|^2 P_b$, where P_b is the peak power of the electron beam. We derive

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a solution to this system with a piecewise defined Green function g,

$$a(\zeta,\tau) = \int_0^{\tau_b} q(\tau')b_0(\tau')g(\zeta,\tau',\tau)d\tau', \qquad (2)$$

and by treating the initial bunching factor b_0 as a random variable we find analytical expressions for the means and variances of various integral quantities. The treatment also produces analytical expressions for amplitude and intensity correlations in the temporal and spectral domains.

The statistics examined in this paper are the means and relative variances of rms duration $\sigma_\tau^2 = \langle\tau^2\rangle - \langle\tau\rangle^2$, rms bandwidth $\sigma_\nu^2 = \langle\nu^2\rangle - \langle\nu\rangle^2$ and pulse energy $\mathcal{E} = \int |a|^2 d\tau$. We use angle brackets to write quantities weighted by the power or spectral density. For expectation values in the probabilistic sense we will use $\mathrm{E}(\cdot)$. For long bunches these statistics are known, there is a detailed review in Ref. [15].

Prior Results for Short Bunches

Bonifacio et al. used the model (1) with an idealized flattop current profile with seeding instead of shot noise to establish the concept of weak superradiance [26]. There have been thorough numerical investigations of the realistic situation with a smooth current distribution and random shot noise by Yurkov et al.

New Results for Short Bunches

We generalize the work on uniform, seeded bunches to incorporate arbitrary current profiles and initial bunching and produce analytical estimates for various statistics of short bunch SASE in the linear regime. The estimates are closed form integrals that involve the current profile q and the Green function g. The integrals are not analytically solvable but easily computed in fractions of a second on a laptop computer. The quantities that emerge from the calculations are slight variants of those we set out to estimate, weighted by pulse energy. As an example, the quantities we can compute that are related to to average bandwidth, the quantities whose mean and variance we can compute are $\mathcal{E}$, $\mathcal{E}\langle\nu\rangle$ and $\mathcal{E}\langle\nu^2\rangle$. By neglecting the correlation with pulse energy, we construct surrogates of the means and variances we are interested in, for example the surrogate for the expectation value of bandwidth is

$$\tilde{\sigma}_\nu = \sqrt{\frac{\mathrm{E}(\mathcal{E}\langle\nu^2\rangle)}{\mathrm{E}(\mathcal{E})} - \frac{\mathrm{E}(\mathcal{E}\langle\nu\rangle)^2 + \mathrm{V}(\mathcal{E}\langle\nu\rangle)}{\mathrm{E}(\mathcal{E})^2 + \mathrm{V}(\mathcal{E})}} \qquad (3)$$

and the surrogate for the expectation value of rms duration is analogous.

The Numerical Code and Dataset

To support the analytical calculations, we used a typical 1D FEL code as defined in Ref. [27]. We simulated 2500 shots for each bunch length with independent shot noise from the model of McNeil [28]. We used a parabolic current profile, $q(\tau) = \{0 \le \tau \le \tau_b :\ 1 - (2\tau/\tau_b - 1)^2,\ 0\}$, the projection of a uniform ellipsoidal 3D distribution. The results are very similar if using a $\sin^2$ or Gaussian profile.

RESULTS, ANALYTICAL AND SIMULATED

An overview of the resulting means and variances is shown in Fig. 1. The statistics are taken at the end of the high gain region, more precisely they were taken at the point of maximum relative variance of pulse energy. Surprisingly, this happens at the same position for all the bunch lengths. These expectations were used for a more fair comparison to the analytical surrogates. When it comes to the mean values of pulse duration and bandwidth, the analogues $\tilde{\sigma}_\nu$ and $\tilde{\sigma}_\tau$ are plotted, for easier comparison with the analytical results. The raw expectations have the same qualitative behaviour.

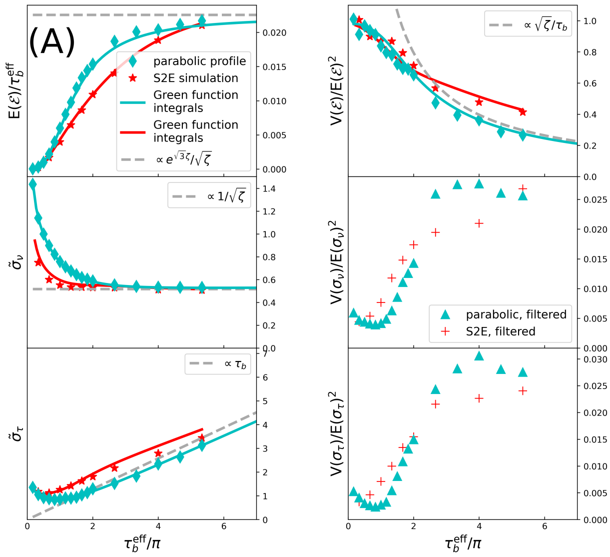

Figure 1: The sample statistics from the numerical dataset (diamonds) and their analytical surrogates (solid lines), as well as asymptotic values for "infinitely long" bunches (dashed lines). Mean values are found in the left column while the corresponding relative variances are found in the right column. Each row corresponds to one quantity of interest: pulse energy, bandwidth and duration respectively. The plots for mean pulse energy are divided by bunch length for clarity's sake. The data for relative variance of bandwidth and duration was filtered for those shots that achieve a pulse energy above the mean, so the analytical equivalents are not shown.

All of the plots have some type of transition through $\tau_b = 3\pi$, corresponding to an effective flattop length of 2π, which Ref. [29] called the length of one SASE spike. One conclusion from the plots is that the relative variance of energy and mean bandwidth contain complementary information about the bunch length: the variance of energy is sensitive at long bunch lengths while the mean bandwidth is sensitive at short bunch lengths. For short bunch lengths, the pulse energy follows an exponential distribution [15], with a significant fraction of shots with negligible energy. To highlight the useful shots, the variance plots for bandwidth and duration use data that was filtered for shots with a pulse energy above the mean ($\approx 37\%$ for an exponential distribution). This filtering precludes any comparison to analytical values so they are not shown. The mean and variance of pulse duration have a stationary point at around $\tau_b = 1.5\pi$, so that the duration statistics are quite stable with respect to fluctuations of the bunch length. Close to this bunch length, the relative standard deviations of bandwidth and duration are quite low, about 5% and the time-bandwidth product is only around 1.5 times the Fourier limit. The conclusion is that there cannot be much variation in the shape of the pulses and it might be feasible to run ultra-fast experiments without direct pulse characterization, as long as the critical bunch length can be achieved reliably. In that case one would only use simple diagnostics for pulse energy and filter out weak shots, the remaining would have almost constant properties. It is convenient that the relative variance of bandwidth is easily measurable during tuning and is a clear indicator of reaching the critical bunch length. For any FEL shot, there is a some radiation that has slipped ahead of the bunch. The length of this "exponential tail" is inversely proportional to the power gain along the undulator. For bunch lengths below the critical, the power gain is slower than exponential so this region of the pulse dominates the duration and peak powers are low. This region of bunch lengths is called weak superradiance [26] and should be avoided when tuning an FEL. To reach shorter pulse durations than the minimum shown here, one needs a more advanced FEL scheme. For example, in the radiation suppression with afterburner setup of Ref. [30–32] completely removes the slippage region of the pulse. We conclude that given sufficient data of the kind shown in Fig. 1, there is enough information to estimate the real bunch length. One has to estimate both the dimensionless bunch length τ_b and the Pierce parameter ϱ, but we have enough independent data points: the mean and variance of bandwidth are suitable for short bunch lengths and the variance of pulse energy provides a sanity check. Supplementary simulations show that reasonable levels of electron energy spread do not impact the results significantly, but that current profiles that are not shaped like a single bump can actually have statistics that differ qualitatively. For example, a sharp spike on top of a longer pedestal is a realistic current profile in some situations. It is no problem to treat these arbitrary current profiles with the same analytical framework.

Overview of the Data

To highlight the relationship between bandwidth, pulse duration, and bunch length, Fig. 2 shows an overview of the numerical data. The figure makes it clear that both data for different bunch lengths and data for different pulse durations overlap significantly. Thus, given one measurement of a large bandwidth, it is not possible to draw any conclusions about either the bunch length or the pulse duration of that shot. It is only with a large sample size of spectra that one can say anything about the bunch length or average duration. Note that even at the optimal bunch length, most of the shots have a longer duration than $\sqrt{2}$ times the Fourier limit. This factor of $\sqrt{2}$ comes from assuming that the pulses are (possibly chirped) Gaussian signals, evidently a bad approximation for SASE pulses. One can also clearly see that for the optimal bunch length, the duration and bandwidth are comparatively tightly clustered, this would be even more apparent if the pulse energy filtering from Fig. 1 was repeated here.

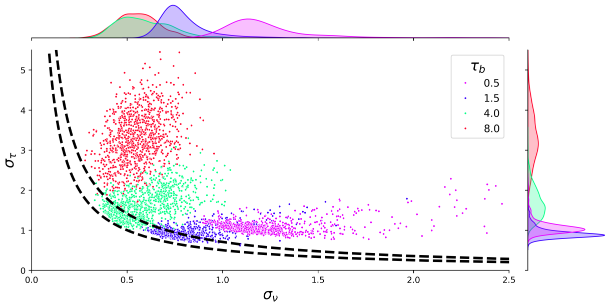

Figure 2: The figure shows scatter plots of the numerical data. The left shows bandwidth against logarithm of pulse energy and the right shows duration against bandwidth. The right plot has a black dashed line for the Fourier limit and a grey dashed line for the heuristic extra factor of $\sqrt{2}$.

SUMMARY

We have incorporated random shot noise and non-uniform current profiles into the classical FEL theory, so that all 1D SASE FEL physics with cold electron bunches in the linear regime are described by a Green function together with the simple statistics of random shot noise. We showed that one can extract significant information from easily acquired statistics and clarify what one can interpret from a single spectrum. In future work we will present the derivations of the theory, use it to calculate short bunch correlation functions and investigate their utility in bunch diagnostics.

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REFERENCES

[1] M. F. Kling and M. JJ. Vrakking, "Attosecond electron dynamics", *Annual Review of Physical Chemistry*, vol. 59, no. 1, pp. 463–492, May 2008. doi:10.1146/annurev.physchem.59.032607.093532

[2] F. Krausz and M. Ivanov, "Attosecond physics", *Reviews of Modern Physics*, vol. 81, no. 1, pp. 163–234, Feb. 2009. doi:10.1103/revmodphys.81.163

[3] M. Nisoli, P. Decleva, F. Calegari, A. Palacios, and F. Martín, "Attosecond electron dynamics in molecules", *Chemical Reviews*, vol. 117, no. 16, pp. 10760–10825, May 2017. doi:10.1021/acs.chemrev.6b00453

[4] The Nobel Committee for Physics, The nobel prize in physics 2023, 2023, https://www.nobelprize.org/prizes/physics/2023/press-release/

[5] Z. Huang and K.-J. Kim, “Review of x-ray free-electron laser theory”, *Physical Review Special Topics - Accelerators and Beams*, vol. 10, no. 3, Mar. 2007. doi:10.1103/physrevstab.10.034801

[6] R. Robles *et al.*, “Hard x-ray attosecond pulses at the lcls”, in *X-Ray Free-Electron Lasers: Advances in Source Development and Instrumentation VII*, p. 14, Jun. 2025. doi:10.1117/12.3058453

[7] E. Prat *et al.*, “Attosecond time-resolved measurements of electron and photon beams with a variable polarization x-band radiofrequency deflector at an x-ray free-electron laser”, *Advanced Photonics*, vol. 7, no. 02, Feb. 2025. doi:10.1117/1.ap.7.2.026002

[8] J. Yan *et al.*, “Terawatt-attosecond hard x-ray free-electron laser at high repetition rate”, *Nature Photonics*, vol. 18, no. 12, pp. 1293–1298, Nov. 2024. doi:10.1038/s41566-024-01566-0

[9] P. Franz *et al.*, “Terawatt-scale attosecond x-ray pulses from a cascaded superradiant free-electron laser”, *Nature Photonics*, vol. 18, no. 7, pp. 698–703, May 2024. doi:10.1038/s41566-024-01427-w

[10] Z. Guo *et al.*, “Experimental demonstration of attosecond pump–probe spectroscopy with an x-ray free-electron laser”, *Nature Photonics*, vol. 18, no. 7, pp. 691–697, Apr. 2024. doi:10.1038/s41566-024-01419-w

[11] J. Duris *et al.*, “Tunable isolated attosecond x-ray pulses with gigawatt peak power from a free-electron laser”, *Nature Photonics*, vol. 14, no. 1, pp. 30–36, Dec. 2019. doi:10.1038/s41566-019-0549-5

[12] YS. Derbenev, AM. Kondratenko, and EL. Saldin, “On the possibility of using a free electron laser for polarization of electrons in storage rings”, *Nucl. Instrum. Methods Phys.*, vol. 193, no. 3, pp. 415–421, Mar. 1982. doi:10.1016/0029-554x(82)90233-6

[13] R. Bonifacio, C. Pellegrini, and LM. Narducci, “Collective instabilities and high-gain regime in a free electron laser”, *Optics Communications*, vol. 50, no. 6, pp. 373–378, Jul. 1984. doi:10.1016/0030-4018(84)90105-6

[14] N. Berrah, J. Cryan, R. Robles, T. Driver, A. Marinelli, and P. Bucksbaum, “Attosecond x-ray sources, methods, and applications at present and future free-electron lasers: tutorial”, *Adv. Opt. Photon.*, vol. 17, no. 3, pp. 623–725, Sep. 2025. doi:10.1364/AOP.540527

[15] M. Yurkov, “Coherence properties of the radiation from x-ray free electron lasers”, *CERN Yellow Reports*, 2018.

[16] G. Ha, K.-J. Kim, J. G. Power, Y. Sun, and P. Piot, “Bunch shaping in electron linear accelerators”, *Rev. Mod. Phys.*, vol. 94, no. 2, p. 025006, May 2022. doi:10.1103/RevModPhys.94.025006

[17] I. Grguraš *et al.*, “Ultrafast x-ray pulse characterization at free-electron lasers”, *Nature Photonics*, vol. 6, no. 12, pp. 852–857, Nov. 2012. doi:10.1038/nphoton.2012.276

[18] N. Hartmann *et al.*, “Attosecond time–energy structure of x-ray free-electron laser pulses”, *Nature Photonics*, vol. 12, no. 4, pp. 215–220, Mar. 2018. doi:10.1038/s41566-018-0107-6

[19] D. Marx, R. Assmann, P. Craievich, U. Dorda, A. Grudiev, and B. Marchetti, “Longitudinal phase space reconstruction simulation studies using a novel x-band transverse deflecting structure at the sinbad facility at desy”, *Nucl. Instrum. Methods Phys. Res. A*, vol. 909, pp. 374–378, 2018. doi:https://doi.org/10.1016/j.nima.2018.02.037

[20] J. W. Yan, H. X. Deng, B. Liu, and D. Wang, “Transverse Deflecting Structure Dynamics for Time-Resolved Machine Studies of Shine”, in *Proc. FEL'19*, Hamburg, Germany, Aug. 2019, pp. 632–634. doi:10.18429/JACoW-FEL2019-THP018

[21] A. A. Lutman *et al.*, “Femtosecond x-ray free electron laser pulse duration measurement from spectral correlation function”, *Phys. Rev. ST Accel. Beams*, vol. 15, no. 3, p. 030705, Mar. 2012. doi:10.1103/PhysRevSTAB.15.030705

[22] S. Serkez *et al.*, “Wigner distribution of self-amplified spontaneous emission free-electron laser pulses and extracting its autocorrelation”, *J. Synchrotron Radiat.*, vol. 28, no. 1, pp. 3–17, Jan. 2021. doi:10.1107/s160057752001382x

[23] P. Catravas *et al.*, “Measurement of electron-beam bunch length and emittance using shot-noise-driven fluctuations in incoherent radiation”, *Phys. Rev. Lett.*, vol. 82, no. 26, pp. 5261–5264, Jun. 1999. doi:10.1103/PhysRevLett.82.5261

[24] EL. Saldin, EA. Schneidmiller, and MV. Yurkov, “Statistical properties of radiation from vuv and x-ray free electron laser”, *Optics Communications*, vol. 148, no. 4, pp. 383–403, 1998. doi:https://doi.org/10.1016/S0030-4018(97)00670-6

[25] S. Krinsky and R. L. Gluckstern, “Analysis of statistical correlations and intensity spiking in the self-amplified spontaneous-emission free-electron laser”, *Phys. Rev. ST Accel. Beams*, vol. 6, no. 5, p. 050701, May 2003. doi:10.1103/PhysRevSTAB.6.050701

[26] R. Bonifacio, F. Casagrande, G. Cerchioni, L. de Salvo Souza, P. Pierini, and N. Piovella, “Physics of the high-gain fel and superradiance”, *La Rivista del Nuovo Cimento*, vol. 13, no. 9, pp. 1–69, Sep. 1990. doi:10.1007/bf02770850

[27] K.-J. Kim, Z. Huang, and R. Lindberg, *Synchrotron radiation and free-electron lasers: principles of coherent x-ray generation*. Cambridge University Press, Mar. 2017. doi:10.1017/9781316677377

[28] B. W. J. McNeil, M. W. Poole, and G. R. M. Robb, “Unified model of electron beam shot noise and coherent spontaneous emission in the helical wiggler free electron laser”, *Physical Review Special Topics - Accelerators and Beams*, vol. 6, no. 7, Jul. 2003. doi:10.1103/physrevstab.6.070701

[29] R. Bonifacio, L. De Salvo, P. Pierini, N. Piovella, and C. Pellegrini, “Spectrum, temporal structure, and fluctuations in a high-gain free-electron laser starting from noise”, *Physical Review Letters*, vol. 73, no. 1, pp. 70–73, Jul. 1994. doi:10.1103/physrevlett.73.70

[30] E. A. Schneidmiller and M. V. Yurkov, “Obtaining high degree of circular polarization at x-ray free electron lasers via a reverse undulator taper”, *Phys. Rev. ST Accel. Beams*, vol. 16, no. 11, p. 110702, Nov. 2013. doi:10.1103/PhysRevSTAB.16.110702

[31] E. A. Schneidmiller, “Application of a modified chirp-taper scheme for generation of attosecond pulses in extreme ultraviolet and soft x-ray free electron lasers”, *Phys. Rev. Accel. Beams*, vol. 25, no. 1, p. 010701, Jan. 2022. doi:10.1103/PhysRevAccelBeams.25.010701

[32] E. Schneidmiller, M. Dreimann, M. Kuhlmann, J. Rönsch-Schulenburg, and H. Zacharias, “Generation of ultrashort pulses in xuv and x-ray fels via an excessive reverse undulator taper”, *Photonics*, vol. 10, no. 6, 2023. doi:10.3390/photonics10060653