A GENERALIZED ANALYTICAL POTENTIAL FOR FAST RFQ BEAM DYNAMICS MODELING

E. Laface*, European Spallation Source ERIC, Lund, Sweden

Abstract

Radio Frequency Quadrupoles (RFQs) are widely used at the front end of linear accelerators to focus, bunch, and accelerate low-energy ion beams. Their electromagnetic field is commonly evaluated with full three-dimensional numerical solvers, which provide accurate field maps but are computationally expensive for fast beam dynamics studies and online applications. The classical Kapchinsky–Teplyakov two-term potential is compact, but its truncated form may not fully reproduce realistic modulated vane fields. In this paper, a generalized analytical RFQ potential is derived from Maxwell's equations in the quasi-static regime relevant to the beam aperture. The scalar potential is obtained as a regular solution of Laplace's equation in cylindrical coordinates and expanded in angular and longitudinal harmonics. The classical two-term approximation is recovered as the lowest-order truncation, while equipotential boundary conditions on the modulated vane surface define a finite matrix problem for the modal coefficients. A framework is also introduced to quantify the accuracy of truncated expansions versus retained terms and transverse radius.

INTRODUCTION

The Radio Frequency Quadrupole (RFQ) is a resonant accelerating structure used at the low-energy front end of ion linacs, where it simultaneously provides transverse focusing, adiabatic bunching, and longitudinal acceleration [1,2]. This is enabled by a four-vane geometry that generates a transverse quadrupole electric field together with a longitudinal component arising from periodic vane modulation.

In RFQ beam dynamics studies, the electromagnetic field is often reconstructed with numerical solvers and then imported into particle tracking codes. Although accurate, this approach can become computationally expensive when repeated field evaluations are required in parameter scans, tuning studies, model-based control, or online applications. For such cases, analytical field representations remain attractive.

The standard analytical description is the two-term potential introduced by Kapchinsky and Teplyakov [1], widely used in RFQ design and simulation codes [3,4]. Because it retains only the lowest-order contributions, it may become insufficient for strongly modulated vanes, for apertures not small compared with the modulation wavelength, or whenever improved field fidelity is required [5].

This work develops a generalized analytical RFQ potential that extends the classical two-term model while preserving computational efficiency. Starting from Maxwell's equations, it derives the scalar potential in cylindrical coordinates and examines how truncation order affects the reconstructed field.

* emanuele.laface@ess.eu

ELECTROMAGNETIC FORMULATION

In vacuum, Maxwell's equations are $\nabla \cdot \vec{E} = 0$, $\nabla \cdot \vec{B} = 0$, $\nabla \times \vec{E} = -\partial \vec{B}/\partial t$, and $\nabla \times \vec{B} = \mu_0 \varepsilon_0 \, \partial \vec{E}/\partial t$. The RFQ is an RF structure, so the full electromagnetic field is time dependent. However, in the beam aperture the transverse dimensions are much smaller than the RF wavelength, and the electric field can be treated to leading order as quasi-electrostatic with harmonic time dependence. We therefore write $\Phi(r,\theta,z,t) = U(r,\theta,z)\cos(\omega t + \phi_s)$, where $U(r,\theta,z)$ is the spatial potential amplitude, $\omega = 2\pi f$ is the RF angular frequency, and ϕ_s is an arbitrary reference phase. To leading order, $\vec{E}(r,\theta,z,t) \simeq -\nabla U(r,\theta,z)\cos(\omega t + \phi_s)$, so the spatial potential satisfies $\nabla^2 U = 0$.

This is the starting point for the analytical RFQ field model considered here.

MODAL EXPANSION IN CYLINDRICAL COORDINATES

The Laplace equation is solved in cylindrical coordinates by separation of variables. We seek solutions of the form $U(r,\theta,z) = R(r)\Theta(\theta)Z(z)$. Using $\nabla^2 = \partial_r^2 + r^{-1}\partial_r + r^{-2}\partial_\theta^2 + \partial_z^2$, substitution into $\nabla^2 U = 0$ gives $\frac{R''}{R} + \frac{1}{r}\frac{R'}{R} + \frac{1}{r^2}\frac{\Theta''}{\Theta} + \frac{Z''}{Z} = 0$.

Angular and Longitudinal Dependence

The angular component must be 2π-periodic. For an RFQ with four-vane symmetry, the natural basis is given by even multipoles, $\Theta_m(\theta) = \cos(2m\theta)$ with $m = 0, 1, 2, \ldots$, and angular eigenvalue $(2m)^2$.

The vane modulation is periodic along the beam axis with fundamental period $\beta\lambda$, where $\beta = v/c$ refers to the synchronous particle and $\lambda = c/f$ is the RF wavelength. The corresponding longitudinal wave number is $k = 2\pi/(\beta\lambda)$, and a convenient basis is $Z_n(z) = \cos(nkz)$, $n = 0, 1, 2, \ldots$, where a sine representation would be equivalent up to a longitudinal phase shift.

Radial Dependence

For a mode indexed by (m,n), the radial equation becomes

$$r^2 R'' + rR' - \left[(nk)^2 r^2 + (2m)^2\right] R = 0. \qquad (1)$$

For $n \geq 1$, Eq. (1) is the modified Bessel equation of order $2m$, with regular solution

$$R_{mn}(r) = A_{mn} I_{2m}(nkr), \qquad n \geq 1, \qquad (2)$$

where I_{2m} is the modified Bessel function of the first kind. For $n = 0$, the radial equation reduces to $r^2R'' + rR' - (2m)^2R = 0$, whose regular solution is $R_{m0}(r) = A_{m0}r^{2m}$.

GENERALIZED RFQ SCALAR POTENTIAL

Collecting the regular angular, longitudinal, and radial modes, the most general scalar potential compatible with the assumed cylindrical periodicity and regularity at the axis can be written as

$$U(r,\theta,z) = \sum_{m=0}^{\infty} A_{m0}\, r^{2m}\cos(2m\theta) + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn}\, I_{2m}(nkr)\cos(2m\theta)\cos(nkz). \quad (3)$$

The full time-dependent potential is $\Phi(r,\theta,z,t) = U(r,\theta,z)\cos(\omega t + \phi_s)$. Equation (3) defines a complete regular modal basis; the actual coefficient pattern is fixed by RFQ symmetries and vane geometry.

SYMMETRY CONSIDERATIONS

The four-vane RFQ geometry implies that adjacent vanes carry opposite potential, while opposite vanes carry the same potential. This enforces the quadrupole condition $U(r,\theta+\pi/2,z) = -U(r,\theta,z)$, which suppresses angular components incompatible with a sign inversion under a $\pi/2$ rotation. The vane modulation also introduces a longitudinal periodicity linked to the RF phase. Equation (3) provides the complete regular basis, while symmetry and boundary conditions select the non-zero coefficients.

RECOVERY OF THE TWO-TERM APPROXIMATION

The classical Kapchinsky–Teplyakov potential is recovered by truncating Eq. (3) to the lowest non-trivial quadrupole term and the first longitudinal harmonic:

$$U(r,\theta,z) \approx \frac{V}{2}\left[A_{10}r^2\cos(2\theta) + A_{01}I_0(kr)\cos(kz)\right], \quad (4)$$

where V is the inter-vane voltage and A_{10}, A_{01} are dimensionless coefficients fixed by the vane geometry. The first term provides the dominant transverse quadrupole focusing, whereas the second produces the longitudinal variation responsible for bunching and acceleration. Equation (4) is thus the lowest-order truncation of the generalized expansion.

BOUNDARY CONDITIONS ON THE MODULATED VANE SURFACE

The modal coefficients are determined by enforcing the equipotential condition on the vane surfaces. Let the vane contour be described by a longitudinally modulated radius, using a simplified cosine parameterization inspired by standard RFQ vane-tip descriptions [6],

$$r_s(z) = a\left[1 + \frac{m_v - 1}{2}\left(1 - \cos\left(\frac{2\pi z}{\beta\lambda}\right)\right)\right], \quad (5)$$

where a is the minimum bore radius and m_v is the modulation factor as in Fig. 1.

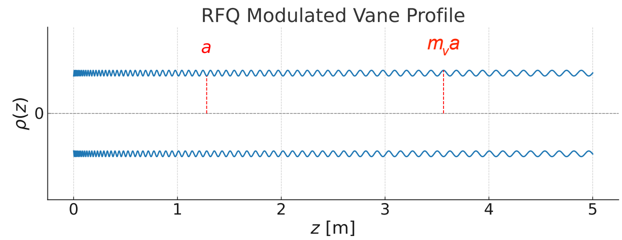

Figure 1: RFQ vane modulation described by Eq. (5).

Choosing the vane at $\theta = 0$ as the positive electrode, the boundary condition is $U(r_s(z),0,z) = V/2$ for all z. Substituting Eq. (3) gives

$$\sum_{m=0}^{\infty} A_{m0}\, r_s(z)^{2m} + \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} A_{mn}\, I_{2m}(nk\, r_s(z))\cos(nkz) = \frac{V}{2}, \qquad \forall z, \quad (6)$$

which is a functional constraint on the unknown coefficients.

FINITE TRUNCATION AND MATRIX FORMULATION

For practical use, the infinite series must be truncated. Let M be the highest angular order and N the highest longitudinal harmonic retained. The truncated potential is

$$U_{MN}(r,\theta,z) = \sum_{m=0}^{M} A_{m0}\, r^{2m}\cos(2m\theta) + \sum_{m=0}^{M}\sum_{n=1}^{N} A_{mn}\, I_{2m}(nkr)\cos(2m\theta)\cos(nkz). \quad (7)$$

Here N_t denotes the total number of retained basis functions; the two-term approximation corresponds to $N_t = 2$. The unknown coefficients are arranged in a vector $\mathbf{a} = (A_{00}, A_{10}, \ldots, A_{M0}, A_{01}, A_{11}, \ldots, A_{MN})^T$. The boundary condition is then enforced either at a discrete set of longitudinal collocation points z_j, or by projection onto a chosen longitudinal basis.

In the collocation approach, for $j = 1, \ldots, P$, one obtains $\mathbf{M}\,\mathbf{a} = \mathbf{b}$, where $M_{j,\alpha} = \Psi_\alpha(r_s(z_j), 0, z_j)$, Ψ_α denotes the α-th truncated basis function, and $b_j = V/2$. If P equals the number of retained modes, the system is square; if P is larger, the coefficients can be obtained in a least-squares sense. This defines a practical route to compute higher-order analytical RFQ coefficients directly from the vane profile.

ELECTRIC FIELD COMPONENTS

Once the scalar potential is known, the electric field follows as $\vec{E}(r,\theta,z,t) = -(\partial U/\partial r\,\hat{r} + r^{-1}\partial U/\partial\theta\,\hat{\theta} + \partial U/\partial z\,\hat{z})\cos(\omega t + \phi_s)$. Equivalently, $E_r = -(\partial U/\partial r)\cos(\omega t + \phi_s)$, $E_\theta = -(1/r)(\partial U/\partial\theta)\cos(\omega t + \phi_s)$, and $E_z = -(\partial U/\partial z)\cos(\omega t + \phi_s)$. Explicit modal expressions follow directly by differentiating Eq. (3).

FIELD ACCURACY AS A FUNCTION OF TRUNCATION ORDER AND RADIUS

The generalized potential of Eq. (3) defines a hierarchy of truncated analytical models. To quantify the effect of truncation, the field reconstructed with a finite number of terms is compared with a reference field obtained from a much richer truncation of the same series.

In the absence of an external numerical field map, an internal high-order reference is introduced by selecting a sufficiently large truncation order N_{ref}. In the present work, the reference field is defined by a normalized solution with $N_{\mathrm{ref}} = 40$, obtained from the same modal expansion and vane boundary condition. Lower-order models with $N_t = 2, 4, 8, 12$ are then constructed by truncating the converged reference expansion.

Let $\vec{E}^{(N_t)}(r,\theta,z)$ denote the electric field reconstructed with N_t retained terms, and let $\vec{E}^{(\mathrm{ref})}(r,\theta,z)$ denote the reference field corresponding to $N_{\mathrm{ref}} = 40$. A convenient shell-averaged relative RMS error, normalized to the reference field energy [7], is

$$\bar{\delta}^{(N_t)}(r_1,r_2) = \sqrt{\frac{\left\langle \left\| \vec{E}^{(N_t)} - \vec{E}^{(\mathrm{ref})} \right\|^2 \right\rangle_{r\in[r_1,r_2],\,\theta,\,z}}{\left\langle \left\| \vec{E}^{(\mathrm{ref})} \right\|^2 \right\rangle_{r\in[r_1,r_2],\,\theta,\,z}}} \times 100, \tag{8}$$

where the average is taken over the selected radial shell, over the angular coordinate, and over one longitudinal modulation period.

For a normalized modulated profile with $ka = 0.8$ and $m_v = 2.0$, Table 1 reports the resulting shell-averaged RMS discrepancy for several truncation orders. As expected, the error decreases as additional terms are retained, with larger discrepancies toward the vane surface.

Table 1: Shell-averaged relative RMS field error $\bar{\delta}^{(N_t)}$ (%) with respect to the internal reference $N_{\mathrm{ref}} = 40$, for $ka = 0.8$ and $m_v = 2.0$.

Range in r/a	$N_t = 2$	$N_t = 4$	$N_t = 8$	$N_t = 12$
$0 \le r/a < 0.2$	31.78	31.78	18.77	8.48
$0.2 \le r/a < 0.4$	16.86	16.86	10.75	5.51
$0.4 \le r/a < 0.6$	13.44	13.25	9.45	5.68
$0.6 \le r/a < 0.8$	15.35	13.25	10.56	7.35
$0.8 \le r/a \le 1.0$	25.30	15.98	13.97	10.64

The first two truncations remain similar over much of the aperture, while the improvement becomes clearer for $N_t = 8$ and $N_t = 12$, especially in the outer radial shells. This shows that a sufficiently rich truncation can serve as an internal benchmark for lower-order RFQ field models without an external field map.

CONCLUSION

A generalized analytical model for the RFQ scalar potential has been derived from Maxwell's equations in the quasi-static regime relevant to RFQ beam dynamics. The model is obtained by solving Laplace's equation in cylindrical coordinates and expanding the regular solution in angular and longitudinal harmonics.

The classical Kapchinsky–Teplyakov two-term potential is recovered as the lowest-order truncation of the generalized expansion. Higher-order coefficients are obtained by imposing equipotential boundary conditions on the modulated vane surface, leading after truncation to a finite linear system for the modal amplitudes.

A high-order internal reference was also introduced to quantify the accuracy of truncated analytical models versus the number of retained terms and transverse radius. The radial error table provides a practical criterion for selecting the truncation order required for a target field accuracy.

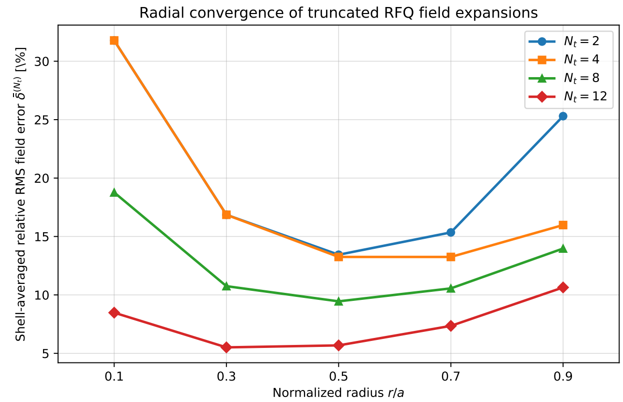

Figure 2: Radial convergence of $\bar{\delta}^{(N_t)}$ versus normalized radius r/a.

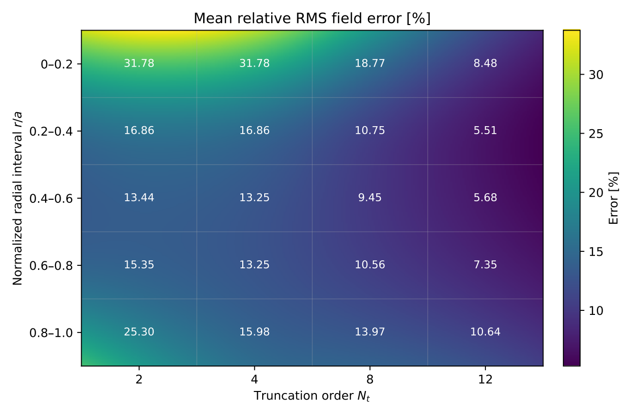

Figure 3: Interpolated heatmap of $\bar{\delta}^{(N_t)}$ versus normalized radial interval and truncation order N_t.

REFERENCES

[1] I. M. Kapchinsky and V. A. Teplyakov, "A Linear Ion Accelerator with Spatially Uniform Quadrupole Focusing," ITEP, Moscow, USSR, 1970.

[2] R. H. Stokes and T. P. Wangler, "Radiofrequency Quadrupole Accelerators and Their Applications," *Annu. Rev. Nucl. Part. Sci.*, vol. 38, no. 1, pp. 97–118, Dec. 1988. doi:10.1146/annurev.ns.38.120188.000525

[3] A. Schempp, “RFQ Design and Performance,” in *Proc. LINAC’90*, Albuquerque, NM, USA, Sep. 1990, pp. 567–569.

[4] K. R. Crandall and T. P. Wangler, “PARMTEQ—A Beam Dynamics Code for the RFQ Linear Accelerator,” Los Alamos National Laboratory, LA-UR-81-2581, 1981.

[5] Z. Zhang *et al.*, “Design of a radio frequency quadrupole for a high intensity heavy-ion accelerator facility,”*Phys. Rev. Accel. Beams*, vol. 25, no. 8, Aug. 2022. doi:10.1103/physrevaccelbeams.25.080102

[6] K. R. Crandall, R. S. Mills, and T. P. Wangler, “Radio-Frequency Quadrupole Vane-Tip Geometries”, in *Proc. PAC’83*, Santa Fe, NM, USA, Mar. 1983, pp. 3554–3557.

[7] T. Chai and R. R. Draxler, “Root mean square error (RMSE) or mean absolute error (MAE)? – Arguments against avoiding RMSE in the literature,”*Geosci. Model Dev.*, vol. 7, no. 3, pp. 1247–1250, Jun. 2014. doi:10.5194/gmd-7-1247-2014