IMPACT OF A QUASI-RESONANT AC DIPOLE EXCITATION ON TRANSVERSE BEAM SPLITTING

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Abstract

To perform Multi-Turn Extraction at the CERN Proton Synchrotron, the beam is transversely split into five separate beamlets, consisting of the core and four islands. It has been shown that an AC dipole excitation effectively controls the characteristics of these beamlets. For this purpose, the AC dipole is set in resonance with the horizontal betatron tune, while the horizontal tune crosses the fourth-order resonance, creating a double-resonance condition, i.e. the simultaneous resonance condition of the horizontal tune and between the horizontal tune and the AC dipole excitation frequency. This increases the fraction of particles that are moved from the core to the islands. Further studies have revealed that slight adjustment of the AC dipole, thereby breaking the double-resonance condition, distributes the beam more evenly across the beamlets. This paper examines this phenomenon by establishing a Hamiltonian model for a system with such a quasi-resonant AC dipole and studying it with numerical simulations.

INTRODUCTION

Since September 2015, the CERN Proton Synchrotron (PS) has routinely delivered high-intensity proton beams to the Super Proton Synchrotron (SPS) using Multi-Turn Extraction (MTE). These beams are used for fixed-target physics at the SPS and generated by utilizing transverse beam splitting, obtained by crossing the horizontal 4^{th}-order resonance [1–4].

The MTE aims to share the beam intensity equally among the five beamlets generated by transverse splitting. Initial experimental studies indicated that the core had higher intensity than the four islands. This was empirically resolved by using the transverse feedback system (TFB) as an AC dipole to excite the beam in the horizontal plane during the resonance crossing process, resulting in an increased beam intensity in the islands.

The impact of this addition has been extensively studied using Hamiltonian models [5, 6], and is currently under further experimental studies to improve the quality of the beam delivered [7, 8]. The results of these studies have revealed features that need to be further investigated theoretically. They point towards a need to better understand the splitting process when the double-resonance condition is broken, specifically when the AC dipole tune is non-resonant.

This work aims to extend previous Hamiltonian models to include this quasi-resonant case and uses detailed numerical simulations to provide an interpretation of the experimental studies that are on-going at the PS [8, 9].

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THE DOUBLE-RESONANCE REGIME

A 1-dimensional Hamiltonian system may be used to model MTE splitting. Let the fractional part of the betatron tune be denoted ν_0, and define $\omega_0 = 2\pi\nu_0$. Furthermore, let the fractional part of the tune of the AC dipole be ν_{AC}, with $\omega_{\text{AC}} = 2\pi\nu_{\text{AC}}$. Based on [6], in a double-resonance condition, i.e. for $\omega_0 \approx \omega_{\text{r}}$ and $\omega_{\text{AC}} = \omega_{\text{r}}$ with $\omega_{\text{r}} = \pi/2$, the Hamiltonian in action-angle coordinates (θ, J) reads

$$H^{\text{dr}}(\theta, J) = \Delta\omega_0 J + \frac{\Omega_2}{2}J^2 + GJ^2\cos(4\theta) + \frac{k_{\text{AC}}}{2}\sqrt{2J}\cos\theta, \tag{1}$$

where $\Delta\omega_0 = \omega_0 - \omega_{\text{r}}$ is proportional to the tune in the chosen frame, which itself co-rotates at ω_{r}. $G \geq 0$ represents the strength of the 4^{th}-order resonance, Ω_2 the detuning, and k_{AC} the strength of the AC dipole. This Hamiltonian admits up to 9 fixed points, up to 5 of which are stable, forming the core and islands shown in Fig. 1, where Cartesian coordinates, $(X, Y) = (\sqrt{2J}\cos\theta, \sqrt{2J}\sin\theta)$, are used.

The solutions of the fixed-point equations are analogous to those of the AC dipole Hamiltonian studied in [5, 10]. Using $\Omega_{2,0} = \Omega_2 + 2G$, $a_0 = \frac{3}{2}^{3/2}\frac{k_{\text{AC}}}{2\Delta\omega_0}\sqrt{-\frac{\Omega_{2,0}}{\Delta\omega_0}}$, and $\Omega_{2,j} = \Omega_2 - 2G$, $a_j = \frac{3}{2}^{3/2}\frac{k_{\text{AC}}(1+\Omega_{2,j}/8G)}{4\Delta\omega_0}\sqrt{-\frac{\Omega_{2,j}}{\Delta\omega_0}}$, they are

$$Y_{k,0} = 0,\ Y_{k,j} = \pm\sqrt{\frac{(X_{k,j}^2(2G - \Omega_{2,j}/2) - \Delta\omega_0}{2G + \Omega_{2,j}/2}} \tag{2}$$

$$\frac{X_{k,0}}{2} = \sqrt{-\frac{2\Delta\omega_0}{3\Omega_{2,0}}}\cos\left(\frac{\arccos a_0 - 2\pi k}{3}\right), k = 0, 1, 2,$$

$$\frac{X_{k,j}}{2} = \sqrt{-\frac{2\Delta\omega_0}{3\Omega_{2,j}}}\cos\left(\frac{\arccos a_j - 2\pi k}{3}\right), j = 1, 2.$$

The position of the islands is proportional to powers of $\Delta\omega_0$, i.e. the islands move as the resonance is crossed by varying ν_0 linearly from ν_0^{start} to ν_0^{end}. Additionally, $\Omega_{2,0}\Delta\omega_0 < 0$ and $\Omega_{2,j}\Delta\omega_0 < 0$ are necessary conditions for the fixed points to be real and hence for the islands to form.

Now, let the splitting efficiency be $\eta = \sum_{i=1}^{4} N_i/(4N)$, where N_i, $1 \leq i \leq 4$ is the number of particles in each of the 4 islands, and $N = \sum_{i=0}^{6} N_i$ is the total number of particles (see Fig. 1 for the definition of phase-space regions).

As in the operational case, assume that the islands form for $\Delta\omega_0 \geq 0$, i.e. $\Omega_{2,j}, \Omega_{2,0} < 0$. In this case, the stable fixed-points corresponding to the islands form as soon as $\nu_0 = 0.25$, and their actions then increase as ν_0 increases. Hence, if the starting betatron tune $\nu_0^{\text{start}} > 0.25$, the islands

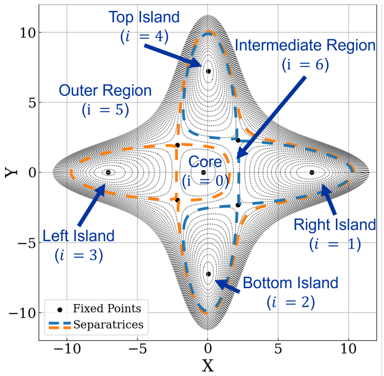

Figure 1: Horizontal phase-space portrait of the model (1) with fixed points (black dots) for $\nu_0 = 0.251$, $\nu_{AC} = 0.25$, and operational MTE parameters. The blue and orange curves are the separatrices. The AC dipole shifts the position of the fixed point at the origin and breaks the symmetry of the shape of the stable islands. The intermediate region, i.e. $i = 6$, vanishes when the horizontal tune increases further.

will form further away from the centre, i.e. away from the location with the highest particle density. Therefore, fewer particles will be captured than if $\nu_0^{\text{start}} \leq 0.25$. Meanwhile, the AC dipole during MTE is turned on exactly when ν_0^{start} is reached and remains active only for a fixed time of 27 ms. Consequently, starting at $\nu_0^{\text{start}} < 0.25$ implies turning the AC dipole off at a lower intermediate ν_0. As there are no islands when $\nu_0 < 0.25$, the AC dipole only negligibly affects the splitting until $\nu_0 = 0.25$ is reached. Hence, the time when it affects the splitting is reduced, which reduces η [11]. Thus, the model implies that the maximum η is reached at $\nu_0^{\text{start}} = 0.25$.

Finally, the AC dipole tune is not a parameter in Eq. (1), hence that model cannot make predictions about how setting $\nu_{AC} \neq 0.25$ would affect η. To study the effect of ν_{AC} on η, a model of the complete PS lattice was used to track a particle bunch starting at various values of ν_0^{start} until the nominal extraction at $\nu_0^{\text{end}} = 0.267$. This corresponds to tracking the particles for 58500 turns (about 123 ms), analogously to what was done in [11] to simulate MTE. The resulting Fig. 2 agrees with experiment, shows complex behaviour of η for $\nu_{AC} \neq 0.25$, and indicates two points to be understood: *i*) For $\nu_{AC} > 0.25$, η retains a close to optimal value of $\eta \in [0.197, 0.203]$, as seen in the upper right part of Fig. 2. *ii*) The maximum η ($\eta_{\max}$) is attained with $\nu_0^{\text{start}} = 0.251$, and not in the double-resonance regime, i.e. $\nu_0^{\text{start}} = 0.25$.

THE QUASI-RESONANT REGIME

Based on the double-resonant case, the quasi-resonant Hamiltonian is obtained by introducing the AC dipole tune $\nu_{AC} \neq 0.25$. In the frame co-rotating with ω_r, this introduces a time dependence in the AC dipole term, resulting in the following Hamiltonian

$$H^{\text{qr}}(\theta, J, t) = \Delta\omega_0 J + \frac{\Omega_2}{2} J^2 + G J^2 \cos(4\theta) + \frac{k_{AC}}{2} \times \sqrt{2J}\,\left(\cos\theta \cos(\Delta\omega_{AC} t) - \sin\theta \sin(\Delta\omega_{AC} t)\right), \tag{3}$$

where $\Delta\omega_{AC} = \omega_r - \omega_{AC} = 2\pi(0.25 - \nu_{AC})$ quantifies the distance of the AC dipole tune from the double-resonance condition.

Although this explicitly time-dependent Hamiltonian no longer permits a simple analysis, one may study its behaviour numerically and through Poincaré maps. The additional time dependence due to the resonance crossing process is parametrised as $\omega_0 = \omega_0(\lambda)$ through a parameter $\lambda(t) \geq 0$.

According to the separatrix-crossing theory in the adiabatic regime [12, 13], a particle in region A_k that crosses a separatrix will be trapped in any particular region A_l that borders that separatrix with probability

$$P_{k\to l} = \max\left\{0, \min\left\{\frac{\mathrm{d}A_l/\mathrm{d}\lambda}{\sum_j \mathrm{d}A_j/\mathrm{d}\lambda}, 1\right\}\right\}, \tag{4}$$

where the summation runs over all regions generated by the separatrix.

This probability multiplied by the local density of the particles in the relevant separatrix, integrated over time, yields the total number of particles that move into that specific region during the entire process. Hence, it provides the splitting efficiency η. Thus, studying the probability of trapping in the islands, $P_{5\to i}$, $1 \leq i \leq 4$, for different values of ν_0 indicates when trapping occurs and how it evolves over time.

The trapping probability has been computed by solving the Hamilton equations related to Eq. (3), and analysing the corresponding Poincaré map. For each region, this allows identification of the trajectory with the largest action remaining trapped in that region, and hence the corresponding area. Doing this for various ν_0 provides the time evolution of the areas and therefore trapping probabilities, since the tune is a function of time during the resonance-crossing process.

Figure 3 shows the results of the numerical calculations: When ν_0 is close to the resonant value, the probability of trapping varies strongly between islands, and even for the same island as a function of ν_{AC}, which demonstrates the impact of the double-resonance regime. When ν_0 is far away from the resonant value, the dependence of the trapping probability on ν_{AC} is largely reduced, and globally, the trapping probability remains constant and equal for all islands. However, at intermediate ν_0, an increase in ν_{AC} causes fewer particles to be trapped in the beginning of the resonance crossing, but more particles to be trapped later. This trend at intermediate ν_0 is visible for all islands, and is not reproducible without the quasi-resonant Hamiltonian in Eq. (3). Hence, the trend shows that the region of $\eta \approx 0.2$ for $\nu_{AC} > 0.25$ in Fig. 2 exists because the trapping probability, integrated throughout the resonance crossing process, does not decrease rapidly for limited increases of $\nu_{AC} > 0.25$. Instead, an increase in ν_{AC} just shifts the maximum of the trapping probability, thus maintaining the total splitting efficiency at a near optimal value.

Note that the absolute values of the trapping probability change between the islands due to the fact that the AC dipole phase is time-dependent [6], and a non-zero initial phase of the AC dipole would produce different trapping probabilities into the individual islands, but not the sum of them.

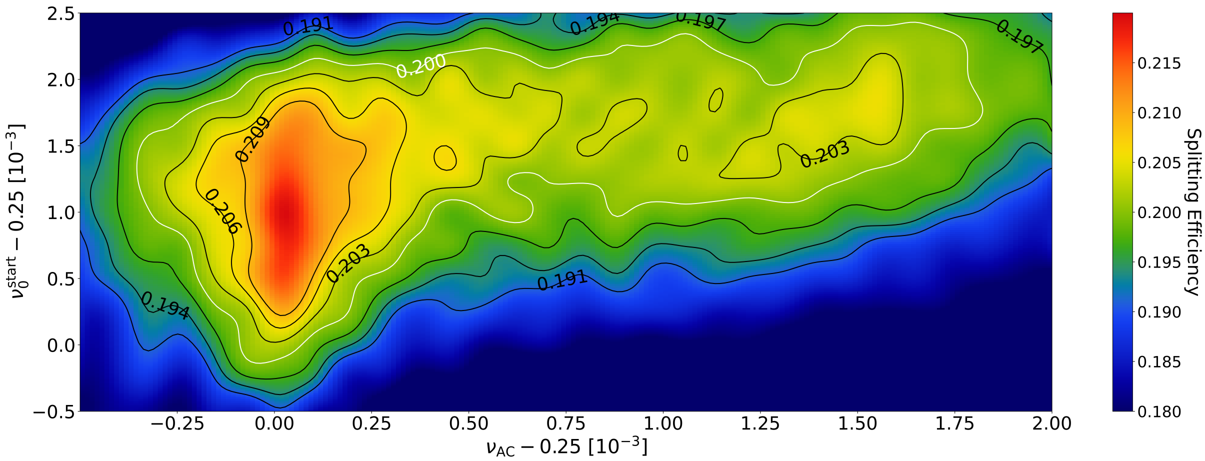

Figure 2: Simulated splitting efficiency η as a function of the AC dipole tune ν_{AC} and initial betatron tune ν_0^{start}. The AC dipole strength is set to $k_{\mathrm{AC}} = 1.5\,\mu\mathrm{rad}$. Lines of constant η are drawn in white for $\eta = 0.20$, and black for $\eta \neq 0.20$.

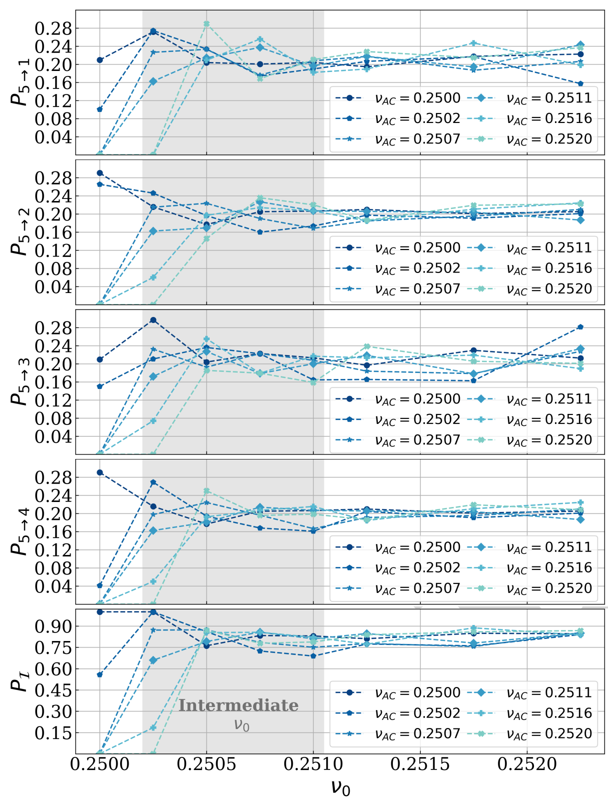

Figure 3: Simulated trapping probabilities for the 4 islands and their sum as a function of ν_0 for various AC dipole ν_{AC} tunes. Intermediate values of ν_0, where results significantly differ from the double-resonant model of Eq. (1), are shaded grey. Note the different vertical scale of the bottom plot, where $P_{\mathcal{T}} = \sum_{i=1}^{4} P_{5\to i}$.

The second question, i.e. why η reaches its maximum at $\nu_0^{\mathrm{start}} = 0.251$, may be answered by considering the change in particle distribution at the time the AC dipole is turned on. Since for $\nu_0^{\mathrm{start}} > 0.25$, the particle distribution when the AC dipole is turned on is already affected by the resonance crossing, the distribution is no longer Gaussian. Thus, setting $\nu_0^{\mathrm{start}} = 0.251$ does not significantly improve the trapping probabilities, as seen in Fig. 3. Rather, this is likely to cause more particles to be close to the separatrices as the AC dipole is turned on. This hypothesis can be verified by simulating MTE analogously to what was done for Fig. 2, but keeping both tunes $\nu_0^{\mathrm{start}} = 0.25$, $\nu_{\mathrm{AC}} = 0.25$ constant. Instead, the time when the AC dipole is turned on, T^{start}, is varied while keeping the total duration for which it is active constant. This separates the effects of just starting the resonance crossing at $\nu_0^{\mathrm{start}} > 0.25$ without the AC dipole, from activating the AC dipole at $\nu_0 > 0.25$. Figure 4 then shows that later activation of the AC dipole causes η to reach η_{max} not at the operational start time T_0^{start}, but 10 ms afterward. This corresponds to a tune at AC dipole activation close to where η_{max} is found in Fig. 2, explaining the observation.

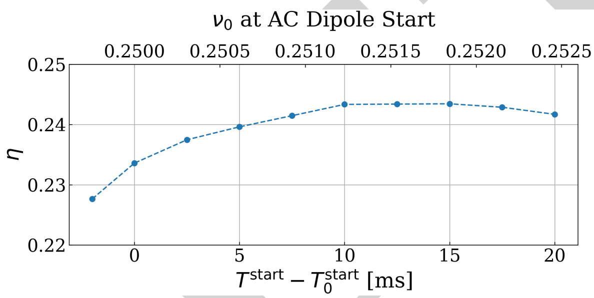

Figure 4: Simulated evolution of the splitting efficiency as the start time of the AC dipole T^{start} is increased from the baseline, operational value T_0^{start}. The top axis shows the betatron tune when the AC dipole is activated.

CONCLUSIONS

The understanding of the MTE process, which includes the use of an AC dipole, has been improved by extending the model to a quasi-resonant one and introducing the AC dipole tune ν_{AC} as an additional parameter. Detailed numerical simulations of both the realistic PS lattice and the extended Hamiltonian model have been carried out to understand the observed behaviour. This knowledge may be used to further optimise MTE operationally and justifies further study of systems in the double-resonance regime or close to it.

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