DYNAMIC APERTURE OPTIMIZATION BASED ON THE CONVEXITY OF THIRD-ORDER RESONANCE DRIVING TERM VARIATIONS

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Abstract

Reducing the longitudinal variation of third-order resonance driving terms (RDTs) is much more effective in enlarging storage ring dynamic aperture (DA) than minimizing third-order one-turn RDTs. Recently, we proved the convexity of the quantitative expression for third-order RDT variations. Then, an efficient numerical method for DA optimization was developed, where a high-quality initial population for an intelligent algorithm is generated with a Gaussian distribution based on this convexity. In this paper, we study the impact of the variance of the Gaussian distribution on the optimization performance of this method. It is found that the method shows good optimization performance for small variances, and that the performance remains robust even at a very small variance. In addition, different intelligent algorithms perform well with a small Gaussian variance.

INTRODUCTION

Dynamic aperture (DA) is a crucial metric in the design of a storage ring, as it influences beam injection and beam lifetime. In the variable space of sextupole strengths, DA typically exhibits a complex distribution characterized by multiple local optima, making its optimization a non-convex optimization problem. Recently, it was demonstrated that reducing the variation of lower-order resonance driving terms (RDTs) along the longitudinal position of a storage ring effectively reduces higher-order nonlinear terms, thereby improving DA much more effectively than reducing the commonly-used one-turn RDTs [1].

Our recent work [2] further proved that the quantitative expression for the longitudinal variation of third-order RDTs, denoted as $f_{3,\mathrm{rms}}$, is a special convex function whose distribution shows strong consistency with that of DA in the sextupole strength space. Therefore, the non-convex DA optimization problem can be approximately treated as a convex optimization problem. Based on this, an efficient DA optimization method was developed [2], in which a high-quality initial population for an optimization algorithm is generated using a Gaussian-distributed sampling based on the geometric structure of $f_{3,\mathrm{rms}}$. In this paper, we will study the impact of the variance of the Gaussian-distributed initial population on the optimization performance of the proposed method. The effect of different optimization algorithms will also be studied.

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CONVEXITY OF THE LONGITUDINAL VARIATION OF THIRD-ORDER RDTS

In this paper, we quantify the longitudinal variation of third-order RDTs using the expression [3]

$$f_{3,\mathrm{rms}} = \sqrt{\sum_{3=j+k+l+m} \sum_{i=1}^{N} |f_{jklm}(z_i)|^2 / N}, \tag{1}$$

where $f_{jklm}(z_i)$ represents the geometric RDT at the position of the i-th sextupole, and N is the number of sextupoles in a storage ring lattice. Recently, we proved that $f_{3,\mathrm{rms}}$ is a special convex function whose iso-surfaces form a series of concentric, coaxial ellipsoidal surfaces in the space of the sextupole strengths [2]. When the constraint of chromaticity correction is included, $f_{3,\mathrm{rms}}$ can be expressed as a quadratic convex function defined on $\mathbb{R}^{N-2}$ [2]:

$$f_{3,\mathrm{rms}} = \sqrt{\frac{1}{2}\mathcal{K}^T \mathcal{D} \mathcal{K} + \mathcal{G}^T \mathcal{K} + \mathcal{H}}, \tag{2}$$

where $\mathcal{K}$ is the variable vector of sextupole strengths, $\mathcal{D}$ is a positive definite matrix, $\mathcal{G}$ is a constant vector, and $\mathcal{H}$ is a constant.

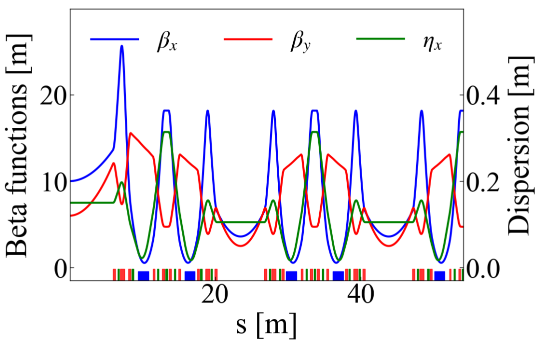

Figure 1: Linear optical functions and magnet layout for half a super-period of the SSRF storage ring lattice.

To illustrate the convexity of $f_{3,\mathrm{rms}}$ and its relationship with DA, the SSRF storage ring lattice [4], shown in Fig. 1, was used for parameter scanning in the sextupole strength space. For each nonlinear solution, $f_{3,\mathrm{rms}}$ was calculated, and DA was tracked through frequency map analysis using the `elegant` code [5]. The total frequency diffusion rate of all surviving particles, denoted as $\sum d_r$, was used to characterize both the DA size and the stability of particle motion. For this scanning case, to reduce computation cost, we regrouped the eight sextupole families into four: two (SF and SD) in the arc sections and the other two (SF1 and SD1) on both sides of the straight sections. In the scanning, the normalized strengths of SF1 and SD1 were free variables over (0, 50) m^{-3} and (-50, 0) m^{-3}, respectively, with the horizontal and vertical chromaticities corrected to (1.0, 1.0) using SF and SD.

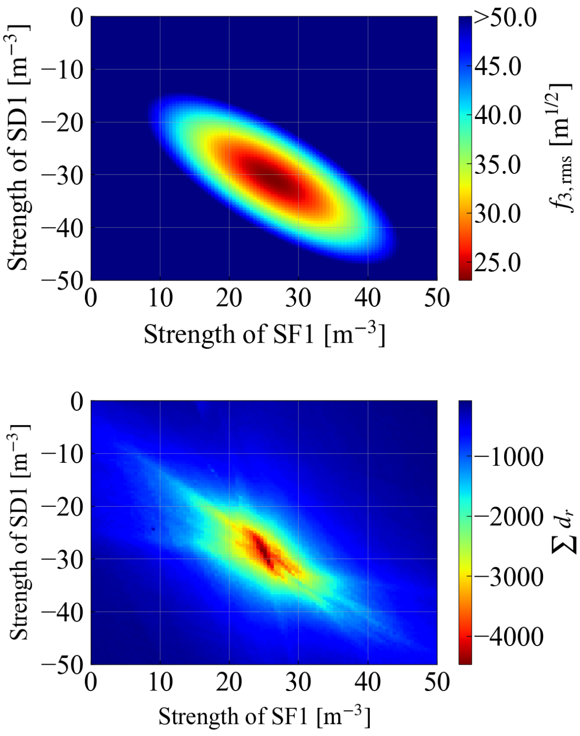

Figure 2: Distributions of $f_{3,\mathrm{rms}}$ (upper) and $\sum d_r$ (lower) in the normalized strength space of the sextupole families SF1 and SD1 for the SSRF lattice with four regrouped sextupole families. In the upper plot, points with $f_{3,\mathrm{rms}}$ values greater than 50.0 are shown in dark blue.

Figure 2 shows the distributions of $f_{3,\mathrm{rms}}$ and $\sum d_r$ in the normalized strength space of the sextupole families SF1 and SD1. It is clearly seen that $f_{3,\mathrm{rms}}$ is a convex function, with its contour lines forming a series of concentric and coaxial ellipses. Moreover, the distributions of $f_{3,\mathrm{rms}}$ and $\sum d_r$ exhibit strong consistency. The regions with lower $\sum d_r$ are continuously distributed within an ellipse with a small value of $f_{3,\mathrm{rms}}$. Since $f_{3,\mathrm{rms}}$ is a convex function and has a strong relationship with DA, it is reasonable to infer that the distributions of $f_{3,\mathrm{rms}}$ and DA will also exhibit good consistency in a high-dimensional variable space with more sextupole families. Therefore, the DA optimization problem can be treated as a roughly approximate convex optimization problem [2].

DYNAMIC APERTURE OPTIMIZATION

Typically, due to insufficient understanding of the distribution of DA, numerical DA optimization methods based on intelligent algorithms usually generate the initial population by uniform random sampling in the variable space. Given the strong consistency between the distributions of $f_{3,\mathrm{rms}}$ and DA in the sextupole strength space, using a Gaussian-distributed sampling based on the geometric structure of $f_{3,\mathrm{rms}}$ to initialize the population can significantly improve the quality of the initial population, thereby greatly enhancing the optimization performance [2]. The population initialization procedure in this method has the following steps: (1) Use some randomly generated nonlinear solutions to fit the parameters $\mathcal{D}$, $\mathcal{G}$, and $\mathcal{H}$ in Eq. (2), and then obtain the central position corresponding to the minimum $f^2_{3,\mathrm{rms}}$; (2) Construct the probability density function of a Gaussian distribution using these parameters, with 3σ equal to λ times the minimum $f^2_{3,\mathrm{rms}}$, where the scaling factor $\lambda > 1.0$; (3) Initialize the population in the sextupole strength space according to this Gaussian distribution.

Impact of Gaussian Distribution Variance

In the initialization procedure, the scaling factor λ is the only parameter to be determined, which is important for optimization performance. A smaller λ helps to search for the global optimum more quickly. However, if λ is too small, corresponding to a very narrow Gaussian distribution, it may degrade the optimization performance. Therefore, it is necessary to study the impact of λ, related to the variance of the Gaussian distribution, on the optimization performance. In this study, the SSRF storage ring lattice with eight sextupole families was used as an example. Its DA was optimized using a differential evolution (DE) algorithm [6] for different initial population cases. In our method, the scaling factor λ was set to 1.005, 1.05, 1.10, 1.25 and 1.50, corresponding to Gaussian-distributed initial populations with increasing variances. For comparison, an initial population based on the uniform random sampling was also generated for DA optimization. In the optimization, the population size was set to 50 with 50 generations, and the optimization objective was to minimize $\sum d_r$.

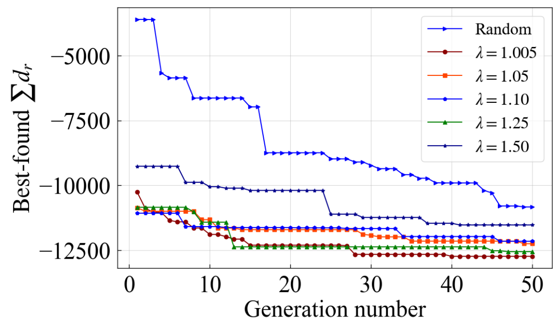

Figure 3: Evolutions of the best-found $\sum d_r$ value with the generation number for initial populations generated by Gaussian-distributed sampling (with five λ values) and uniform random sampling.

Figure 3 shows the convergence curves of the best-found $\sum d_r$ for populations initialized using the Gaussian-distributed sampling (with five λ values) and uniform random sampling. It is seen that compared to the uniform random sampling, Gaussian-distributed sampling based on the convexity of $f_{3,\mathrm{rms}}$ significantly improves the quality of the initial population and accelerates the convergence. For the four Gaussian-distributed initial populations with $\lambda \leq 1.25$, although their variances differ, their convergence speeds and optimization results are similar. Moreover, even for a very narrow Gaussian distribution corresponding to $\lambda = 1.005$, the optimization performance does not degrade. This is due to the algorithm's inherent exploration capability and the strong consistency between the distributions of $f_{3,\mathrm{rms}}$ and DA. This indicates that when the variance of the Gaussian-distributed initial population is small (e.g., $\lambda \leq 1.25$), our method shows good optimization performance.

Effect of Different Optimization Algorithms

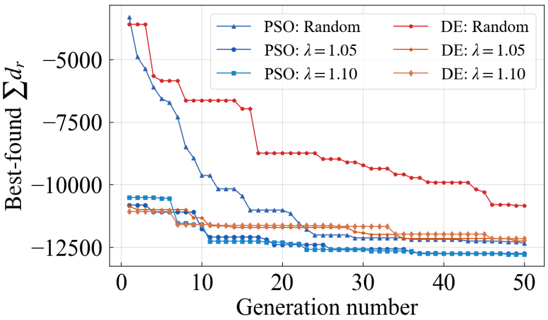

Figure 4: Evolutions of the best-found $\sum d_r$ value for DE and PSO with three initial populations generated by Gaussian-distributed sampling with $\lambda = 1.05$ and $\lambda = 1.10$, and by uniform random sampling.

To study whether other intelligent algorithms still perform well when the variance of the Gaussian distribution is small, we also employed the particle swarm optimization (PSO) algorithm [7] in our method. The DA of the SSRF lattice was optimized using PSO with initial populations generated by Gaussian-distributed sampling with $\lambda = 1.05$ and $\lambda = 1.10$, and by uniform random sampling. Figure 4 shows the convergence curves of the best-found $\sum d_r$ for the DE and PSO algorithms with three initial populations. It is seen that when using PSO, compared with the uniform random sampling, the Gaussian-distributed sampling still significantly improves the optimization performance. We also see that for a small variance of the Gaussian distribution, the PSO algorithm also achieves good optimization performance. In addition, for these two kinds of samplings, PSO has faster convergence than DE. The DAs of the optimal solutions obtained using the two algorithms with Gaussian distributions are shown in Fig. 5.

CONCLUSION

Reducing the longitudinal variation of third-order RDTs is very effective in enlarging DA, and its quantitative expression $f_{3,\mathrm{rms}}$ is a special convex function. Based on this convexity, an efficient DA optimization method was recently developed, where a Gaussian-distributed initial population for an intelligent algorithm is generated according to the geometric structure of $f_{3,\mathrm{rms}}$. Using the SSRF storage ring

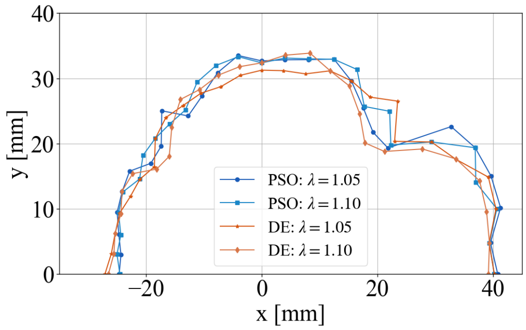

Figure 5: DAs of the optimal solutions obtained using the DE and PSO algorithms with Gaussian-distributed initial populations with $\lambda = 1.05$ and $\lambda = 1.10$.

lattice as an example, this paper further studied the impact of the variance of the Gaussian distribution on the optimization performance of this method. It was found that this method exhibits good optimization performance when the variance is small, and the performance does not degrade even when the variance is very small. Moreover, different intelligent algorithms perform well with a small variance of the Gaussian distribution.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China under Grant No. 12375325.

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