EXACT SOLUTION OF DUAL HARMONIC MOTION

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Abstract

The pendulum equation is the archetype for longitudinal motion within an RF bucket. The restoring force is sinusoidal. The case that the confining potential is composed of fundamental and second harmonic is called "dual harmonic". The solution to the pendulum equation of motion in terms of Jacobi elliptic functions is well known, for 140 years. The solution to the dual harmonic equation, also in terms of elliptic functions, is almost unknown. Here we present the solutions for confined and unbounded motion, with almost arbitrary ratio of the two harmonics. The oscillation frequency as a function of oscillation amplitude is a collateral prediction.

INTRODUCTION

Let t, x and $p = \dot{x}$ be time, position and momentum; and dots denote time derivatives ($dx/dt \equiv \dot{x}$). This article concerns the motion of a particle having the equation of motion $\ddot{x} + \omega^2 F(x) = 0$. The force law $F(x)$ and potential function $U(x)$, respectively, are

$$\begin{aligned} F(x) &= \sin x \ + \ q \sin 2x \\ U(x) &= [1 - \cos x] \ + \ (q/2)[1 - \cos 2x]\,. \end{aligned} \tag{1}$$

q is the ratio of harmonic amplitudes. Positive ratio $q > 0$ increases the restoring force yielding the bunch shortening mode. Negative ratio $q < 0$ reduces the restoring force, providing the bunch lengthening mode. Ensembles of particles that homogeneously populate equi-density contours of the corresponding Hamiltonian $H(x, \dot{x})$, constitute bunches. The nature of the fixed points changes abruptly at $|q| = 1/2$. For $|q| < 1/2$, there is a single well with stable fixed point (SFP) at $x = 0$ and unstable fixed points (UFP) at $x = \pm\pi$. For $|q| > 1/2$, a second, local well emerges; and the nature of the fixed points (FPs) change. For the bunch lengthening case, $q = -1$, Shaposhnikova wrote a solution for the motion in the dual-harmonic well and used it as basis for the calculation of the beam transfer function [1]. In Oct/Nov 2023 Koscielniak obtained the solution for $q = +1$, and in Jan/Feb 2024 extended the method to arbitrary q. A comprehensive investigation is reported in Ref. [2].

Basic Properties of Jacobi Elliptic Functions

The elliptic functions may be defined either by differential equations or as the inverse of incomplete integrals. There are four sets of 3 functions, giving a total of twelve. However, the properties of all may be derived from the principal set: cn, sn, dn. Each function has two arguments (z, m); z is the phase in radian, and m is the Jacobi parameter. $k = \sqrt{m}$ is called the elliptic modulus. The Jacobi elliptic functions are solutions of the system of equations and initial conditions

$$\dot{X} = YZ, \qquad \dot{Y} = -ZX, \qquad \dot{Z} = -mXY\,. \tag{2}$$

$$\mathrm{sn}(0, m) = X(0) = 0, \quad \mathrm{cn}(0, m) = Y(0) = 1 \tag{3}$$

and $\mathrm{dn}(0, m) = Z(0) = 1$.

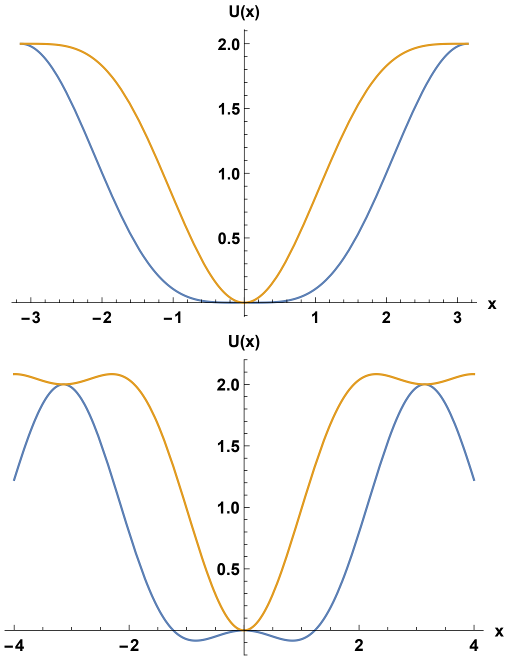

Figure 1: Potential $U(x)$ versus position x. Upper: $|q| = 1/2$. Lower: $|q| = 3/4$. Negative and positive q shown blue and yellow, respectively.

DUAL HARMONIC WELL

We write the potential function in terms of $x/2$; leading to the Hamiltonian:

$$\begin{aligned} 2J\omega^2 &= \omega^2 U(x) + (1/2)(\dot{x})^2 \\ U(x) &= 2[\sin(x/2)]^2 \left\{1 + 2q - 2q[\sin(x/2)]^2\right\}. \end{aligned} \tag{4}$$

The solution is of the form:

$$\sin[x(t)/2] = Y(t)/\sqrt{1 - X^2(t)}. \tag{5}$$

Substitution into the Hamiltonian yields: $2J\omega^2 =$

$$2\omega^2 Y^2 \left[\frac{(1+2q)}{B} - 2q\frac{Y^2}{B^2}\right] + 2\frac{[B\dot{Y} + YX\dot{X}]^2}{AB^2}\,. \tag{6}$$

$$A = (1 - Y^2 - X^2)\,, \quad B = (1 - X^2)\,. \tag{7}$$

We substitute the explicit functions

$$Y = \sqrt{r}\,\mathrm{cn}(at, m) \quad \text{and} \quad X = \sqrt{p}\,\mathrm{sn}(at, m) \tag{8}$$

where a, r, p, m, J are adjustable constants; leading to a cubic equation in powers of $[\mathrm{sn}(at, m)]^2$. The coefficients of

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all powers must be zero; and this leads to a set of four simultaneous nonlinear equations to be solved for the adjustable constants. It follows that $0 < r \leq 1$ and $0 < p \leq 1$.

The *Mathematica* symbolic manipulation software finds solutions of these equations; and we may choose which of the triplet m, p, r is the independent parameter. The number of solutions is reduced by imposing the constraints $0 < m \leq$, $0 < r \leq 1$, $0 < p \leq 1$, $J > 0$ and $a > 0$. There are four types of solution: (i) libration with $p = r$; (ii) rotation with $r = 1$ and $p \neq 1$; (iii) solutions with $p = p(r)$ or $r = r(p)$; and (iv) trivial solutions corresponding to the particle being stuck at a fixed point. *Libration* means inside the RF bucket. *Rotation* means outside the RF bucket.

Fundamental Solutions

The underlying properties of cn, sn, dn imply that there are two fundamental solutions: $p \to r$ (libration) and $r \to 1$ (rotation). Other solutions are accidental duplicates for special conditions between p and r. This implies that we can short-cut the whole procedure by introducing $p = r$ or $r = 1$ at an early stage of the calculation. In this way, we dispose of the irrelevant solutions; and work with simper algebraic equations. When $p = r$, the sextic reduces to a quadratic in powers of sn^2; and when $r = 1$ the sextic reduces to a cubic in powers of sn.

Libration & Rotation, $q = -1/2$

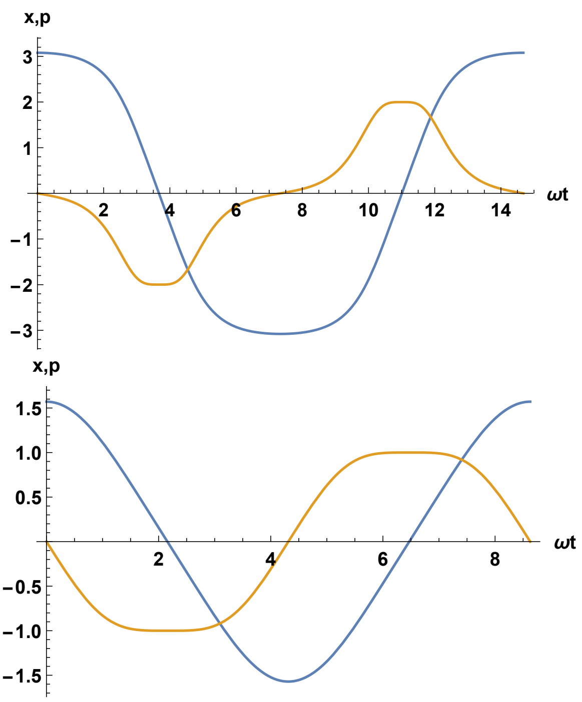

Figure 2: Libration: Position (blue) and momentum (yellow) trajectories vs phase. Upper: $r = 0.999$. Lower: $r = 0.5$

Whereas the phase space portraits Figs. 3 & 5 could have been constructed from the Hamiltonian by solution for $\dot{x}$ in terms of x and J, the trajectories $x(t)$ and $\dot{x}(t)$ versus time

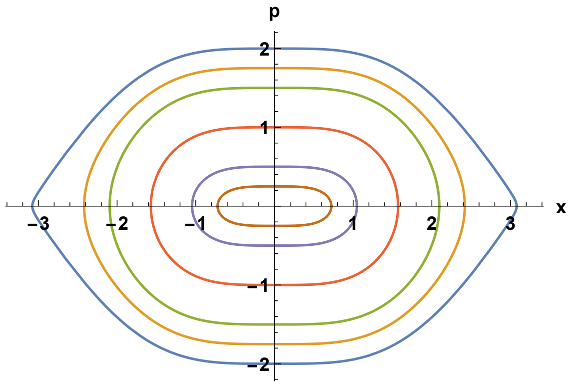

Figure 3: Contours of constant Hamiltonian for libration in bunch-lengthening mode. Blue: $r = 0.999$; gold: $r = 7/8$; green: $r = 3/4$; coral: $r = 1/2$; purple: $r = 1/4$; brown: $r = 1/8$. Abscissa: position (radian). Ordinate: momentum (dimensionless).

can only be graphed if explicit solutions to the equation of motion have been obtained.

Treat r as Parameter

There are two distinct *libration* solutions. The most useful and least constrained is

$$\begin{aligned} a &= \sqrt{1 + 2q(1-2r)}\,\omega, \\ m &= [r(1-2qr)]/[1+2q(1-2r)], \quad p \to r, \\ J &= r + 2qr(1-r)\,. \end{aligned} \tag{9}$$

This solution is valid over all q, provided r is chosen appropriately:

$$\begin{aligned} &q \leq -\frac{1}{2} \quad \& \; 0 < r < 1 + \frac{1}{2q} \quad \text{or} \\ &-\frac{1}{2} < q \leq +\frac{1}{2} \quad \& \; 0 < r < 1 \quad \text{or} \\ &q > \frac{1}{2} \quad \& \; 0 < r < \frac{1}{2q}\,. \end{aligned}$$

The Hamiltonian is $H = 2J\omega^2$ and rotation period given by $\tau = 4K(m)/a$.

Treat p as parameter

There are three distinct solutions. The first has a imaginary when $q = +1/2$ and the third has $a = 0$ whe $q = -1/2$, signaling issues with the behaviour at the SFPs. The first is the *rotation* solution:

$$\begin{aligned} a &= \sqrt{p - 2q}\,\omega/p, \\ m &= p(p - 4q + 2pq)/(p - 2q), \\ r &= 1, \qquad J = (p - 2q + 2pq)/p^2\,. \end{aligned} \tag{10}$$

This solution is valid over the ranges:

$$\begin{aligned} q < 0 \quad &\& \quad 0 < p < 1 \quad \text{or} \\ 0 < q < 1/2 \quad &\& \quad 4q/(1+2q) < p < 1\,. \end{aligned}$$

The Hamiltonian is $H = 2J\omega^2$ and rotation period given by $\tau = 2K(m)/a$.

Libration, $q = +1/2$

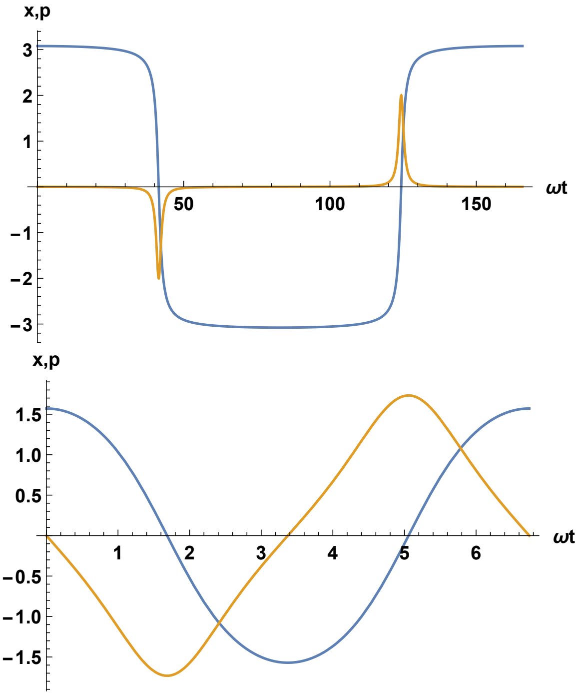

Figure 4: Libration: Position (blue) and momentum (yellow) trajectories versus phase. Left: $r = 0.999$. Right: $r = 0.5$

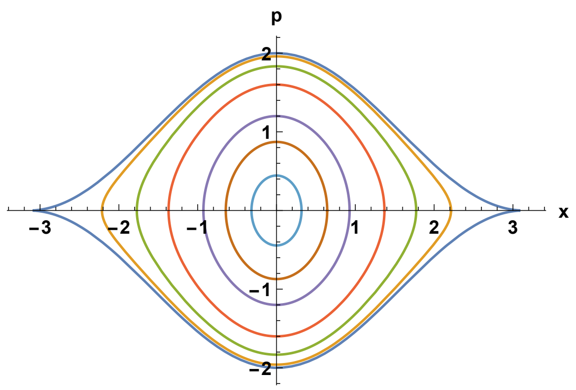

Figure 5: Contours of constant Hamiltonian for libration in bunch-shortening mode. Blue: $r = 0.999$; gold: $r = 0.8$; green: $r = 0.6$; coral: $r = 0.4$; purple: $r = 0.2$; brown: $r = 0.1$; light blue: $r = 0.025$. Abscissa: position (radian). Ordinate: momentum (dimensionless).

AMPLITUDE RATIO $q > 1/2$

When the amplitude ratio satisfies $q > 1/2$, a hanging valley appears above the main valley floor; as is shown in the lower portion of Fig. 1. We move the centre of the coordinate system to $x = \pm\pi$ in order to symmetrize the potential with respect to the local SFP within the hanging valley. In the new coordinates, the fixed points move and the mathematical structure of the energy equation is modified. Therefore, we cannot assume to find all the relevant solutions merely by the substitutions $p = r$ or $r = 1$. (Such a manoeuvre fails to find the libration in the hanging valley.) So we must perform the complete analysis.

The potential takes on the form:

$$U(x) = 2 \; - \; 2(1+2q)[\sin(x/2)]^2 \; + \; 4q[\sin(x/2)]^4 \, .$$

We insert U into the Hamiltonian and make the substitution $\sin(x/2) \to Y/\sqrt{1-X^2}$ to find

$$2\omega^2 A\left[\frac{1}{B} + 2q\frac{Y^2}{B^2}\right] \; + \; 2\frac{[B\dot{Y} + Y X \dot{X}]^2}{A\,B^2} \;=\; 2J\omega^2 \, . \quad (11)$$

We multiply throughout by AB^2 and insert explicit forms for $Y(t)$ and $X(t)$. There results an equation in powers of $\mathrm{sn}(at,m)$; and their coefficients must all be zero, leading to four simultaneous equations.

We may solve for the adjustable constants in terms of r or p or m. Note, thus far, q may take on any value.

Treat r as parameter There are three distinct solutions. The first is a trivial solution. The second is inconsistent with constraints. The third is the *libration* solution for oscillations inside the (hanging valley) secondary potential well:

$$\begin{aligned}
a &= \sqrt{1+2q(r-1)}\sqrt{r-1}\,\omega, \\
m &= \frac{r(1+2qr)}{[1+2q(r-1)](r-1)}, \\
p &= \frac{-2qr}{1+2q(r-1)}, \quad J = (1-r)(1+2qr)\,.
\end{aligned}$$

This solution is valid for the range $q > +1/2$ and $0 < r < (2q-1)/(4q)$.

Treat p as arameter There are three distinct solutions. The first is a re-write of the $q < 0$ *rotation* solution (with respect to the shifted coordinate) for bunch lengthening; valid for $q < 0$ and $0 < p < 2q/(2q-1)$.

CONCLUSION

We have presented the exact analytic solution for longitudinal motion in a dual RF system with fundamental $h =$ and first harmonic $h = 2$. We gave expressions for the motion inside (libration) and outside (rotation) the RF bucket. The ratio of harmonic amplitudes is quite extensive, and satisfies $|q| < 1$. Concerning rotation, we have not found formulae for the rotary motion when the amplitude ratio exceeds $q = 1/2$. A challenge to the reader is to find a valid trial function for motion that traverses the bunch-shortening potential well for $q > +1/2$.

REFERENCES

[1] Elena Shaposhnikova, "Bunched beam transfer matrices in single and double rf systems", CERN, Geneva, Switzerland, CERN-SL-94-19 (RF), Aug. 1994.

[2] Shane Koscielniak, "Motion in Dual-Harmonic Potential Well", TRIUMF, Vancouver, Canada, TRIUMF Beam Note TRI-BN-24-04R.