NONLINEAR SOLUTION OF THE LANDAU-LIFSCHITZ-GILBERT EQUATION

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Abstract

Ferrites may be used to shrink cavity resonator size. If a variable magnetic bias field is applied, the resonance frequency may be varied to follow proton/ion revolution frequency. The RF magnetic field can be applied either parallel or perpendicular to the bias. If the ferrite is near saturation, the incremental permeability is dominated by the electron spins - which are described by the Landau-Lifschitz-Gilbert (LLG) equation. Here (for the first time) the LLG equation is solved analytically in the strong nonlinear regime to give dependence of permeability on RF amplitude. For RF parallel to bias, a constant magnetization emerges. For RF perpendicular to bias, the resonance frequency rises with amplitude leading to the hardening oscillator "jump" instability.

SPIN-MAGNETIZATION EQUATION

The Landau-Lifshitz [1] and Gilbert [2] versions of the spin relaxation equation of motion are mathematically equivalent, although they differ in the interpretation of the relaxation parameter α. $\mathbf{H}$ will be the effective magnetic field, and $\mathbf{M}$ the magnetization. We write the time derivative $d\mathbf{M}/dt \equiv \dot{\mathbf{M}}$. Here $g \equiv \mu_0\gamma_e$. The LLG equation is:

$$\dot{\mathbf{M}} = g\mathbf{H}\times\mathbf{M} \; + g\frac{\alpha}{M}\,\mathbf{M}\times(\mathbf{H}\times\mathbf{M}). \tag{1}$$

The LLG equation admits steady-state solutions, but does not generate them. For the steady-state, we must apply the equations of magneto-statics. We assume the bias field is sufficiently strong to align all the magnetic domains into a single domain; so the material is near/at saturation. Because of the demagnetizing field, both the DC and RF fields inside the material differ from the outside applied values. We denote static values by upper case letters, and dynamic values by lower case.

The demagnetizing field is $\mathbf{H}_d = \bar{\mathbf{N}}\mathbf{M}$ with magnetization $\mathbf{M}$ and tensor $\bar{\mathbf{N}}$. However, $\mathbf{M}$ depends on the prior history of the external applied $\mathbf{H}_2$ which penetrates into the interior. The DC internal field is $\mathbf{H} = \mathbf{H}_2 + \mathbf{H}_d = \mathbf{H}_2 \; - \bar{\mathbf{N}}\mathbf{M}$.

In order to find the dynamics, we add an AC/RF term $\mathbf{h}_2$ to the external H-field. This will apply an alternating torque to the spins, causing them to precess. Summing over all the spins leads to a gyrating magnetic moment $\mathbf{m}$. The AC internal field is $\mathbf{h} = \mathbf{h}_2 - \bar{\mathbf{n}}\mathbf{m}$, where $\bar{\mathbf{n}}$ is the AC/RF demagnetization tensor. Hence, with respect to the LLG equation, we make the substitutions:

$$\mathbf{M}\to\mathbf{M}+\mathbf{m} \quad\text{and}\quad \mathbf{H}\to(\mathbf{H}_2-\bar{\mathbf{N}}\mathbf{M})+(\mathbf{h}_2-\bar{\mathbf{n}}\mathbf{m})\;. \tag{2}$$

The resulting equation contains a large number of terms; and nonlinearity up to cubic terms. This complexity is dealt with by taking single-component forms for $\mathbf{H}_2$ and $\mathbf{h}_2$; and assuming that the coordinate axes are taken parallel with the symmetry axes of the magnetized object, in which case, the matrices $\bar{\mathbf{N}}$ and $\bar{\mathbf{n}}$ take diagonal form. Thus we take a Cartesian system and the DC bias aligned with the z-axis such that $\mathbf{H}_2 = [0,0,H_z]$. We assume the bias field is strong, so the magnetic material is near saturation with domains aligned. The static magnetization is $\mathbf{M} = [0,0,M_z]$. For brevity, we define $\tilde{\mathbf{H}} \equiv \mathbf{H}_2 \; - \bar{\mathbf{N}}\mathbf{M} \; - \bar{\mathbf{n}}\mathbf{m}$.

RF PARALLEL TO BIAS

We consider the case that an external RF magnetic field is applied parallel to an external DC magnetic bias field. The angular frequency of the RF field is ω. $\mathbf{h} = [0,0,h_z\cos(\omega t)]$ and $\mathbf{m} = [m_x(t), m_y(t), m_z(t)]$. The combination $g\times H$ has the dimensions of angular frequency.

A simplification is needed to make the system of equations tractable. We take the case of a sphere or cube with equal demagnetizing factors $N_1 = N_2 = N_3$ and $n_1 = n_2 = n_3$. However, we admit the possibility that AC and DC factors are not identical; and that $n_1 \neq N_3$. $\mathbf{H}$ and $\mathbf{M}$ are substituted in the Landau Eq. (1), leading to equations of motion for the three components of magnetization:

$$\begin{aligned}
\dot{m}_x &= -g(\tilde{H}_z + h_z\cos\omega t)[Mm_y + \alpha m_x(M_z+m_z)]/M\\
\dot{m}_y &= +g(\tilde{H}_z + h_z\cos\omega t)[Mm_x - \alpha m_y(M_z+m_z)]/M\\
\dot{m}_z &= +g\alpha(\tilde{H}_z + h_z\cos\omega t)(m_x^2+m_y^2)/M.
\end{aligned} \tag{3}$$

We make the definitions $(\omega/g)R \equiv \tilde{H}_z$ and $(\omega/g)r \equiv h_z$; and $s = \omega t$ and $m' \equiv dm/ds$. When $R = 1$, the drive frequency becomes $\omega = gH_z$ which can be considered the resonance frequency. The ratio $r/R = h_z/\tilde{H}_z$. This is the most economical notation; but if we wish to keep the fields H_z, h_z fixed while the drive frequency is varied, then R and r must be updated.

The magnitude of the magnetization is preserved by the LLG equations, and so we may replace $(m_y^2+m_x^2)$ by $M^2-m_z^2$ in the equation for $\dot{m}_z$. Equation (3) is solved for the time-varying magnetizations:

$$\begin{aligned}
m_x(s) &= \operatorname{sech}[Mc_1 \; + \alpha\chi][c_2\cos\chi - c_3\sin\chi]\\
m_y(s) &= \operatorname{sech}[Mc_1 \; + \alpha\chi)][c_3\cos\chi + c_2\sin\chi]\\
m_z(s) &= M\tanh[Mc_1 \; + \alpha\chi] \; - M_z.
\end{aligned} \tag{4}$$

Here $\chi(s) = R\,s + r\sin s$. The constant of integration c_1 is chosen to yield the initial value of m_z. Thus $\tanh(Mc_1) = m_z(0)/M$. At large times $m_z \to (M - M_z)$ and $m_x, m_y \to 0$. The reader may recognize that $\cos(r\sin s)$ and $\sin(r\sin s)$ are representations of the Bessel functions, and may be Fourier analysed into a spectrum with frequency components at multiples of ω (the drive frequency). The form of

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m_x, m_y indicates a precession of the spin, with fundamental frequency proportional to H_z/g and modulations whose amplitudes are related to the RF drive h_z.

RF PERPENDICULAR TO BIAS

We consider the case that an external RF magnetic field is applied perpendicular to an external DC magnetic bias field. The DC bias aligned with the z-axis such that $\mathbf{H}_2 = [0, 0, H_z]$. The angular frequency of the RF field is ω. $\mathbf{h} = [0, h_y \cos(\omega t), 0]$ and $\mathbf{m} = [m_x(t), m_y(t), m_z(t)]$.

Linearized Solution

The solution of the linearized LLG equation is well known as the Polder [3,4] tensor. But in fact, Polder himself references Kittel [5,6] as the originator; and, moreover, neither author considered a damping term in the form of a triple-vector product. We write the drive as $h_y(t) = h_y \cos(\omega t)$. We define $H_p \equiv H_z + (n_1 - N_3)M_z$ and $H_q \equiv H_z + (n_2 - N_3)M_z$; and $M_z \equiv \rho M$. The linearized LLG equation is:

$$\begin{aligned}\dot{m}_x &= g(h_y M_z \cos\omega t \; - H_p m_x \alpha\rho \; - H_q m_y)\\ \dot{m}_y &= g H_p m_x + g\alpha\rho(h_y M_z \cos\omega t \; - H_q m_y)\\ \dot{m}_z &= 0\,. \end{aligned} \tag{5}$$

The solution is the sum of a complementary function, which decays, and a particular integral that is the steady state. The gyromagnetic resonance condition is $\omega^2 = g^2 H_p H_q$. When driven at that specific frequency, the response is:

$$\begin{aligned}m_x(t) &= \frac{h_y M \cos\omega t}{(H_p + H_q)\alpha}\\ m_y(t) &= \frac{h_y M_z \cos\omega t}{(H_p + H_q)} + \frac{h_y M \sin\omega t}{(H_p + H_q)\alpha}\sqrt{\frac{H_p}{H_q}}\\ m_z &= c_3 \quad \text{is constant.}\end{aligned} \tag{6}$$

Evidently, the in-phase component of m_x and the quadrature component of m_y are boosted by $1/\alpha$. Note both m_x and m_y are perpendicular to the bias; h_y drives precession in the x, y plane; as would an applied RF component h_x.

Nonlinear Solution for Sphere

The linearized equations of motion (5), and their solution, do not conserve the magnitude of the magnetization. The full nonlinear LLG equation is so complicated that single-component forms for $\mathbf{H}_2 = [0, 0, H_z]$, $\mathbf{M} = [0, 0, M_z]$ and $\mathbf{h} = [0, h_y(t), 0]$, and diagonal forms for $\bar{\mathbf{N}}$ and $\bar{\mathbf{n}}$, are insufficient to make the problem tractable. Additional symmetries are needed.

In the case of the sphere or cube $n_1 = n_2 = n_3$, and assuming the DC and AC factors are equal $n_1 = N_3$. $\mathbf{H}$ and $\mathbf{M}$ and $\mathbf{m} = [m_x(t), m_y(t), m_z(t)]$ are substituted in the Landau Eq. (1), leading to equations of motion for the three components of magnetization:

$$\begin{aligned}\dot{m}_x &= \mu\left[-(M m_y \; + \alpha m_x U_z) H_z - (\alpha m_x m_y - M U_z)\, h_y \cos\omega t\right]\\ \dot{m}_y &= \mu\left[+(M m_x \; - \alpha m_y U_z) H_z \; + \alpha\,(m_x^2 + U_z^2)\, h_y \cos\omega t\right]\\ \dot{m}_z &= \mu\left[+\alpha\,(m_x^2 + m_y^2) H_z \; - (M m_x + \alpha m_y U_z)\, h_y \cos\omega t\right].\end{aligned}$$

Here $\mu \equiv (g/M)$ and $U_z(t) \equiv M_z + m_z(t)$. These equations do not have an exact analytic solution. However, the method of *harmonic balance* may find an approximate Fourier series steady-state solution. Let $s \equiv \omega t$. We take the trial solution:

$$\mathbf{m}(t) = [A_x \cos(s + p_x), A_y \sin(s + p_y), m_0 + A_z \sin(2s + p_z)].$$

Here A_j are adjustable amplitudes and p_j are adjustable phases, both to be determined; and the index $j = x, y, z$. Note that we have included a DC term, m_0, and a frequency doubling in the third component $m_z(t)$.

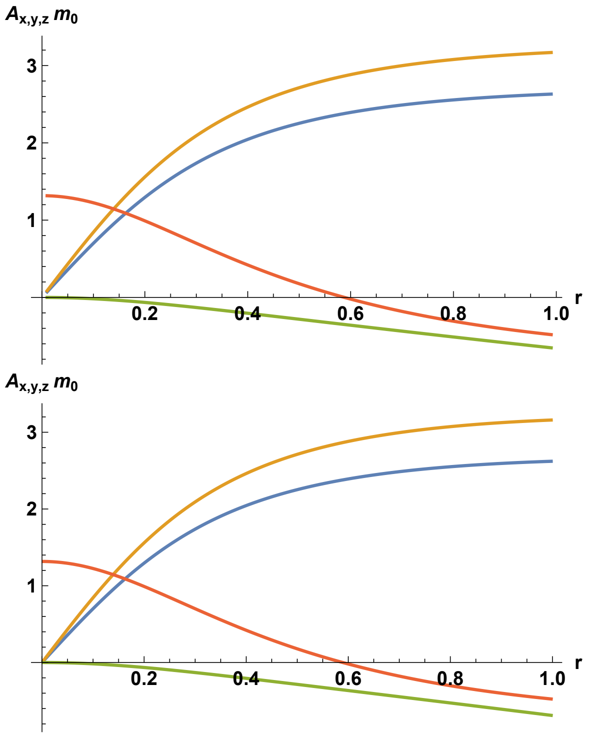

Figure 1: Magnetization components (A_x, A_y, A_z, m_0) versus drive amplitude r. Blue = A_x, gold = A_y, green = A_z, coral = m_0. Resonance frequency ($R = 1.2$) above drive frequency. Upper: values found by harmonic balance. Lower: values from numerical simulation.

The exact solution conserves the magnitude M^2. However, the trial solution may introduce periodic modulations – which must be minimized. Hence in addition to the three equations for $\dot{m}_j$, there is a constraint equation $[\mathbf{M} + \mathbf{m}(t)]^2 = M^2$. We call this the *zeroth* equation. The trial form is substituted into the zeroth and three LLG Equations; we label this set 0, 1, 2, 3 respectively. The four equations are Fourier analyzed. Let $[\cos s, \sin s]$ be fundamental components; and $[\cos ns, \sin ns]$ be n^{th} harmonics. We label the n^{th} harmonic cosine and sine components of Equation number j as C_{jn} and S_{jn} respectively. The trial form has seven (7) adjustable parameters. There are more equations than unknowns. The strategy is to minimize error in the lowest harmonics. Hence, we solve the system of equations

$$C_{00} = C_{11} = S_{11} = C_{21} = S_{21} = C_{32} = S_{32} = 0, \tag{7}$$

for the vector $\mathbf{x} \equiv [m_0, A_x, p_x, A_y, p_y, A_z, p_z]$ of unknowns. They are solved numerically, by Newton-Raphson iteration, for given values of the parameters M, M_z, H_z, h_y, α. The most economical notation is that the quantities $g \times H_z$ and $g \times h_y$ be written in terms of the drive frequency ω. Thus $gH_z \equiv \omega R$ and $gh_y \equiv \omega r$. Evidently $r/R = h_y/H_z$; and this ratio is a measure of the degree of nonlinearity. When $R < 1$, the resonance frequency is below the drive frequency; $R = 1$ corresponds to resonance; when $R > 1$, the resonance frequency lies above the drive frequency.

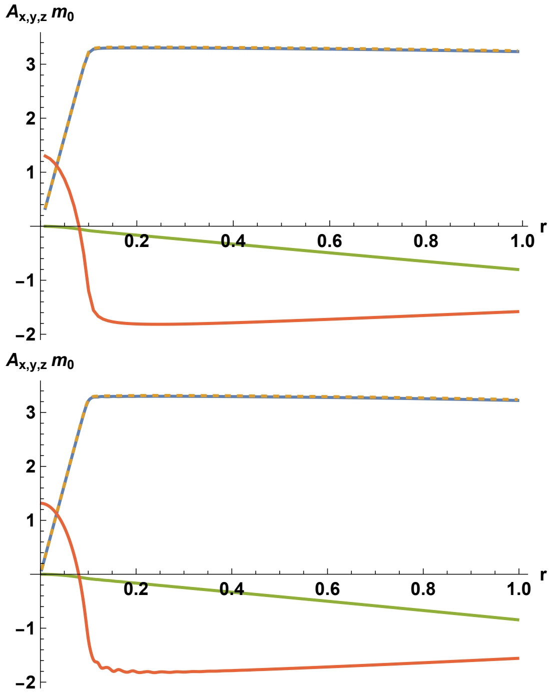

Figure 2: Magnetization components (A_x, A_y, A_z, m_0) versus drive amplitude r. Blue = A_x, dashed gold = A_y, green = A_z, coral = m_0. Resonance frequency ($R = 1.0$) equal to drive frequency. Upper: values found by harmonic balance. Lower: values from numerical simulation.

The behaviour of the system is investigated by setting R fixed, and then systematically varying r over the range $[0, 1]$. The process is repeated for $R > 1$, $R = 1$ and $R < 1$. Typically, A_x, A_y initially rise linear with r, but then plateau[1] at surprisingly small values of r/R. Contrastingly, A_z initially rises quadratic with r, but then rises linear. m_0 initially varies quadratic, then changes sign (opposes M_z), then varies steeply with r, and then plateaus. In the plateau[2] region of r, all amplitudes vary slowly and linearly. These predictions have been confirmed by direct numerical solution of the LLG equations for the same parameter values. In all cases, the Gilbert damping $\alpha = 0.05$.

CONCLUSION

Figures 1 and 2 compare analytic values of A_x, A_y, A_z, m_0 against those extracted from numerical simulation for the

[1] "Plateau" is here used as a verb.
[2] "Plateau" is here used as an adjective.

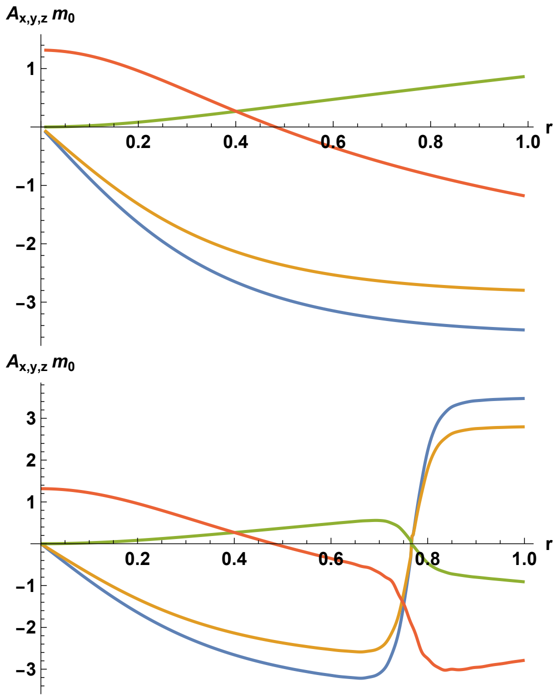

Figure 3: Magnetization components versus drive amplitude r. Blue = A_x, gold = A_y, green = A_z, coral = m_0. Resonance frequency ($R = 0.8$) below drive frequency. Upper: values found by harmonic balance. Lower: values from numerical simulation.

cases $R = 1.2$ and $R = 1.0$ respectively. When $R = 1$ the amplitudes A_x and A_y are indistinguishable and therefore become overlayed. In both cases, the analytic solutions provide good predictions of the response across the entire range $r = [0, 1]$. Figure 3 compares predictions for the magnetization amplitudes when ($R = 0.8$) the drive frequency is above the resonance frequency; the agreement is good upto $r = 0.8$, but diverge thereafter.

The reader will recall that solution of the linearized LLG Equation showed the components m_x, m_y to be proportional to h_y, and to be boosted at resonance. In the nonlinear theory m_x, m_y saturate at surprisingly low values of h_y/H_z; and m_z (which is zero in the linear theory) slowly increases parallel to the magnetic bias field. This has a simple explanation: there is a finite quantity of spin-sites available to be gyrated by the RF drive; when they are all used up, there can be no further increase in the magnetization $\mathbf{m}$.

Additional cases and further details of these phenomena are presented in Ref. [7].

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