GROUND-STATE AND RF-FREQUENCY SCALING FOR SUPERCONDUCTING QUANTUM SYSTEMS*

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Abstract

A fixed-volume comparison of three-dimensional hard-wall quantum confinement is extended to a volume-dependent frequency model for superconducting quantum systems. The equivalent confinement-frequency scale is obtained from the lowest Dirichlet eigenvalue and compared with radio-frequency (RF) scales used in superconducting radio-frequency (SRF) cavities. The calculation separates the Schrödinger confinement scale from the electromagnetic wavelength scale: the quantum frequency follows an inverse two-thirds volume law, whereas a dimensional RF half-wave scale based on $L = V^{1/3}$ follows an inverse one-third volume law. Equating the two frequency scales maps MHz-to-GHz SRF operation to submicron-to-micron local confinement lengths, supporting applications to superconducting islands, surface defects, quasiparticle traps, and cavity-integrated quantum devices.

INTRODUCTION

Geometry-dependent quantum confinement provides a compact way to connect boundary shape to the lowest allowed kinetic energy. A fixed-volume comparison of rectangular, cylindrical, spherical, and prolate ellipsoidal hard-wall domains shows that the sphere gives the lowest eigenvalue among the representative compact shapes considered [1]. This result follows the standard quantum-mechanical connection between the Dirichlet Laplacian eigenvalue and the particle-in-a-box ground state [2, 3].

This paper reformulates that result for accelerator and superconducting-quantum-system applications. A superconducting cavity has an RF resonant frequency fRF, whereas the quantum confinement model gives a ground-state energy-equivalent frequency scale $f_q = E_1/h$. These frequencies have different physical origins: f_{RF} is an electromagnetic eigenfrequency, whereas f_q is a Schrödinger/Laplacian kinetic-energy benchmark. The comparison is therefore a scale map rather than a replacement for an electromagnetic cavity eigenmode calculation.

The present model does not describe the electromagnetic eigenmodes of SRF cavities themselves, but instead provides a local confinement-scale estimate for microscopic superconducting regions interacting with RF fields. The useful comparison is therefore not to treat the full cavity as an infinite potential well. Instead, the model identifies the local volume scale at which a confined electron-like or Cooper-pair-like degree of freedom would have a frequency comparable to the applied RF field. This scale is relevant to superconducting surface oxides, hydride or impurity inclusions, weak links, quasiparticle traps, nanoscale islands, and engineered quantum-computing structures.

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FIXED-VOLUME SCALING MODEL

Inside a hard-wall domain the non-relativistic ground state is obtained from the Dirichlet Laplacian eigenvalue problem [2, 3]. The hard-wall boundary condition imposes zero wave function on the boundary of the domain, so the first eigenvalue sets the kinetic-energy scale.

$$-\frac{\hbar^2}{2m_{\text{eff}}}\nabla^2\psi = E\psi \tag{1}$$

$$-\nabla^2\psi = \lambda\psi, \qquad E_1 = \frac{\hbar^2}{2m_{\text{eff}}}\lambda_1 \tag{2}$$

For geometrically similar domains, scaling all lengths by a factor s changes the volume as s^3 and the eigenvalue as s^{-2}. Therefore, the lowest eigenvalue can be written with a unit-volume shape factor, separating pure size dependence from shape dependence.

The different scaling exponents arise because the Schrödinger kinetic-energy operator scales as L^{-2}, whereas electromagnetic wavelength scales as L^{-1}.

$$\lambda_1(g,V) = C_g V^{-2/3} \tag{3}$$

$$f_q(g,V,m_{\text{eff}}) = \frac{E_1}{h} = \frac{h}{8\pi^2 m_{\text{eff}}} C_g V^{-2/3} \tag{4}$$

For an electron-like particle, the effective mass is the electron mass. For a simple Cooper-pair estimate, the effective mass is twice the electron mass, so all quantum frequencies are reduced by one half. With $h = 6.62607015 \times 10^{-34}$ J s and $m_e = 9.1093837 \times 10^{-31}$ kg, the electron-mass constant is

$$K_e = \frac{h}{8\pi^2 m_e} = 9.2125 \times 10^{-6}\ \text{m}^2/\text{s} \tag{5}$$

GEOMETRY FACTORS

For a cube of side L, the ground-state eigenvalue is obtained by adding the three equal Cartesian contributions. Since the volume is L^3, the cube shape factor is 29.6088. For a sphere of radius R, the lowest eigenvalue is obtained from the first spherical Bessel root. This gives the spherical factor 25.6463, used as the reference minimum.

$$C_{\text{sph}} = \pi^2\left(\frac{4\pi}{3}\right)^{2/3} = 25.6463 \tag{6}$$

For a cylindrical infinite well, the ground-state eigenvalue contains the first zero of the Bessel function

J_0, α_{01} = 2.4048255577. With x = H/R, the ground-state eigenvalue is

$$\lambda_1 = \frac{\alpha_{01}^2}{R^2} + \frac{\pi^2}{H^2} \tag{7}$$

Minimization of the unit-volume shape factor gives x = $\sqrt{2}\pi/\alpha_{01}$ = 1.8475, or R/H = 0.541. The optimized-cylinder factor is 28.0164. The resulting shape factors and frequency ratios are summarized in Table 1.

Table 1: Unit-Volume Shape Factors and Frequency Ratios

Geometry	C_g	C_g/C_{sph}
Sphere	25.6463	1.000
Optimized cylinder	28.0164	1.092
Cube	29.6088	1.155

Thus, at fixed volume, the optimized cylinder has a 9.2% higher confinement-frequency scale than the sphere, and the cube has a 15.5% higher frequency. Shape changes are systematic but much smaller than frequency changes produced by volume variation over many orders of magnitude.

FREQUENCY RANGE

For the spherical electron-mass case, the product of the electron-mass constant and spherical shape factor is 2.3627 × 10^{-4} m²/s. The frequency range follows directly from Eq. (8).

$$f_{q,\mathrm{sph}} = 2.3627 \times 10^{-4} V^{-2/3} \tag{8}$$

Each increase of volume by a factor of 10^9 lowers the quantum frequency by a factor of 10^6. The comparison with a representative RF half-wave scale uses $L = V^{1/3}$, so this dimensional RF scale varies as $V^{-1/3}$. This RF line is only a dimensional scale and not an SRF cavity eigenmode. Representative numerical values are listed in Table 2, and the corresponding log-log trend is shown in Fig. 1.

$$f_{\mathrm{RF}} \approx \frac{c}{2L} = \frac{c}{2V^{1/3}} \tag{9}$$

Table 2: Spherical Quantum Frequency and Dimensional RF Half-Wave Scale

V	f_q (m_e)	f_q ($2m_e$)	RF scale
1 nm³	236 THz	118 THz	150 PHz
1 µm³	236 MHz	118 MHz	150 THz
1 mm³	236 Hz	118 Hz	150 GHz
1 cm³	2.36 Hz	1.18 Hz	15.0 GHz
1 m³	236 µHz	118 µHz	150 MHz

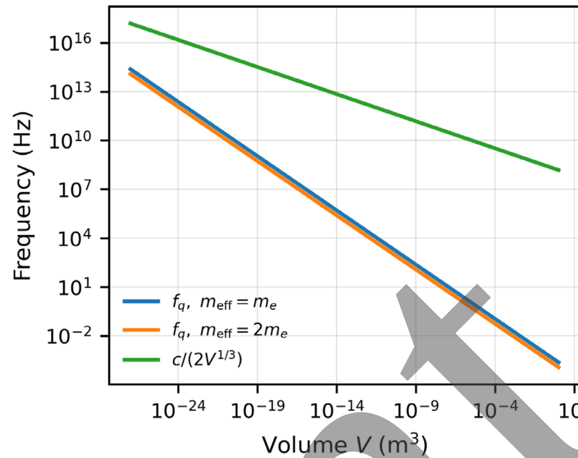

Figure 1: Spherical quantum frequency and dimensional RF scale versus volume. The quantum curves use Eq. (8), while the RF curve uses Eq. (9) with $L = V^{1/3}$.

CONNECTION TO SRF CAVITY RF FREQUENCY

SRF accelerator cavities are driven at an RF frequency chosen by electromagnetic design and beam-dynamics requirements [4]. Representative superconducting-cavity frequencies include 81.25 MHz, 162.5 MHz, and 325 MHz classes used in the RAON RF system [5, 6], as well as 650 MHz and 1.3 GHz classes in other SRF applications [7]. The free-space wavelength and half-wavelength scales are calculated independently from Eq. (10), and the resulting values are summarized in Table 3.

$$\lambda_{\mathrm{RF}} = \frac{c}{f_{\mathrm{RF}}} \tag{10}$$

Table 3: RF Wavelength and Half-Wavelength Scales

f_{RF}	λ_{RF}	$\lambda_{RF}/2$
81.25 MHz	3.690 m	1.845 m
162.5 MHz	1.845 m	0.922 m
325 MHz	0.922 m	0.461 m
650 MHz	0.461 m	0.231 m
1.3 GHz	0.231 m	0.115 m

To map an SRF frequency onto the microscopic quantum scale, set $f_q = f_{RF}$. For a spherical confinement region, Eq. (11) gives the equivalent cubic-root length $L_{eq} = V_{eq}^{1/3}$, and Eq. (12) gives the corresponding equivalent volume. The resulting values are summarized in Table 4.

$$L_{\mathrm{eq}} = \left[\frac{K_e (m_e/m_{\mathrm{eff}}) C_{\mathrm{sph}}}{f_{\mathrm{RF}}}\right]^{1/2} \tag{11}$$

$$V_{\mathrm{eq}} = L_{\mathrm{eq}}^3 \tag{12}$$

For example, at 325 MHz and electron mass, Eq. (11) gives L_{eq} = 853 nm and Eq. (12) gives V_{eq} = 6.20 × 10^{-19} m³. For the Cooper-pair estimate, the coefficient is reduced by one half and the length becomes 603 nm. For a true sphere, the tabulated L_{eq} is a volume-equivalent cubic-root length rather than a radius.

Table 4: Equivalent Cubic-Root Length Giving RF Frequency Matching

f_{RF}	L_{eq} (m_e)	L_{eq} ($2m_e$)
81.25 MHz	1.71 μm	1.21 μm
162.5 MHz	1.21 μm	0.853 μm
325 MHz	0.853 μm	0.603 μm
650 MHz	0.603 μm	0.426 μm
1.3 GHz	0.426 μm	0.301 μm

APPLICATION TO SUPERCONDUCTING GROUND STATES

In superconductors, the ground state is normally described by a macroscopic condensate phase rather than by a single-particle wave function. Nevertheless, geometry enters through boundary conditions, mode spectra, coherence length, penetration depth, surface participation, and the effective electromagnetic environment [4, 8]. The hard-wall model provides a simplified kinetic-energy reference that can be compared with more complete superconducting descriptions.

The scale separation in Tables 3 and 4 is the central result for SRF use. The electromagnetic RF wavelength is macroscopic, typically centimeters to meters, while the quantum confinement length giving the same numerical MHz-to-GHz frequency is submicron to micron scale. Therefore, the proposed application is strongest for localized superconducting regions rather than the full cavity volume.

When local f_q approaches the applied f_{RF}, scale matching suggests that RF fields may more efficiently probe or perturb localized modes, provided that electromagnetic participation and material coupling are present. When f_q is far above f_{RF}, the confined state is high in energy and effectively frozen on the RF time scale. When f_q is far below f_{RF}, circuit and electromagnetic boundary conditions dominate, and the state may contribute to loss, noise, or quasiparticle dynamics.

RELEVANCE TO QUANTUM COMPUTING

Superconducting quantum-computing hardware uses microwave resonators and qubits whose frequencies are engineered mainly by Josephson energy, charging energy, capacitance, inductance, and electromagnetic mode structure [9]. The present hard-wall model is not a replacement for a qubit Hamiltonian. It is instead a scale-matching estimate for residual quasiparticle traps, confined surface states, package modes, and small superconducting volumes that can interact with microwave fields.

For transmon-like or cavity-integrated devices, the model can help determine whether a nearby confined superconducting or quasiparticle region has an energy scale comparable to the readout or control frequency. The frequency and length scales in Tables 2-4 quantify this comparison. The model also separates pure volume scaling from shape anisotropy: the sphere is the minimum-frequency reference, while the cube and optimized cylinder are shifted upward by their geometric factors.

CONCLUSION

A fixed-volume ground-state energy model has been generalized into a volume-dependent frequency map and connected to SRF cavity RF frequency. The key result is the inverse two-thirds volume scaling in Eq. (4), while the RF scale is electromagnetic and follows a different length dependence. Equating the two frequency scales identifies an equivalent microscopic confinement length rather than the physical size of the full cavity. For 81.25 MHz to 1.3 GHz SRF frequencies, the corresponding hard-wall cubic-root lengths are approximately 0.3–1.7 μm for electron-like and Cooper-pair-like masses. The model should therefore be used as a compact scale estimate for local superconducting regions, surface physics, accelerator cavities, and quantum-computing structures. The main numerical results are collected in Tables 1–4 and illustrated in Fig. 1.

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