ONE POSSIBLE APPROACH FOR INTRABEAM SCATTERING CALCULATION OF ARBITRARY PHASE SPACE DISTRIBUTIONS

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Abstract

Traditional analytical models for intrabeam scattering (IBS) typically assume a Gaussian beam distribution in phase space. In this paper, we present an extended analytical IBS model that employs Hermite-Gaussian polynomials as basis functions to calculate the IBS diffusion coefficient for arbitrary phase space distributions. The generating function method is adopted to simplify the relevant calculations into a numerically solvable integral form. This approach retains the efficiency inherent to analytical models while ensuring calculation accuracy for non-Gaussian beam distributions.

INTRODUCTION

Advanced accelerator-based light sources demand increasingly high brightness and coherence, requiring smaller emittance and shorter bunch lengths. In this regime, intrabeam scattering effect (IBS) becomes a critical performance-limiting factor.

Traditional IBS analytical models, such as those of Piwinski [1] and Bjorken–Mtingwa [2], assume a Gaussian phase space distribution. However, advanced concepts like steady-state microbunching (SSMB) can lead to significantly non-Gaussian distributions [3]. While Monte Carlo methods can handle such cases [4], they are time-consuming and offer limited theoretical insight for design. This paper presents a more analytical approach for calculating IBS effects for arbitrary phase space distributions.

Diffusion Coefficient Due to IBS Effect

Early IBS studies focused on the growth rate τ, as IBS is a slow, multi-turn process. However, recent research on SSMB storage rings shows that even single-pass IBS kicks on momentum spread can significantly affect performance. Therefore, we extend the IBS description from τ to the diffusion coefficient in Fokker-Planck equation (FPE).

IBS effect corresponds to a diffusion process in 6D phase space, described by the FPE [5]. The diffusion coefficient tensor D_{ij} quantifies this process. From a microscopic perspective, the Langevin equation provides an equivalent description [6], allowing random kicks to be generated according to D_{ij}. This technique is particularly useful when such kicks are critical, such as the SSMB scenario. Thus, adopting the diffusion coefficient is a natural extension.

In general, the diffusion coefficient is a local function of phase space coordinates. To simplify the FPE, it is often averaged over the entire phase space, becoming a constant [6]. This approximation is valid when IBS effect is weak enough compared to the external forces, so that the phase space distribution remains dominated by single-particle motion. Currently, this condition is still valid in most cases.

Under this simplification, D_{ij} is approximated by its average $\langle D_{ij}\rangle$.

$$\langle D_{ij}\rangle = \left\langle \frac{d(\delta p_i \delta p_j)}{dt} \right\rangle, \tag{1}$$

where δp_i refers to the components of momentum change $\delta \mathbf{p}$ in a single collision event.

Following the similar deduction of Bjorken and Mtingwa [2]:

$$\begin{aligned} \langle D_{ij}\rangle = \frac{1}{2N} \iiint \rho_1(\mathbf{x}, \mathbf{p}_1)\rho_2(\mathbf{x}, \mathbf{p}_2)(q_i q_j) d^3\mathbf{x} d^3\mathbf{p}_1 d^3\mathbf{p}_2 \\ \int \frac{m d^3\mathbf{p}'_1}{E'_1} \int \frac{m d^3\mathbf{p}'_2}{E'_2} \\ |M_c|^2 \frac{m^2}{E_1 E_2} \frac{\delta^4(p'^{\mu}_1 + p'^{\mu}_2 - p^{\mu}_1 - p^{\mu}_2)}{(2\pi)^2}, \end{aligned} \tag{2}$$

$$\mathbf{q} := \mathbf{p}'_1 - \mathbf{p}_1 = (q_1, q_2, q_3).$$

Equation (2) is the general form of the simplified diffusion coefficient. Further simplifications depend on the assumed phase space distribution.

If $\rho(\mathbf{z})$ is Gaussian, Eq. (2) is equivalent to the Bjorken–Mtingwa result. For double- or multi-Gaussian distributions, it yields results from our unpublished work [7]. Next, we present an approach to evaluate $\langle D_{ij}\rangle$ for arbitrary phase space distributions using Hermite–Gaussian polynomials.

Hermite-Gaussian Polynomials

Hermite-Gaussian polynomials are a class of basis functions in $\mathbb{R}^n$ space. Their definition is well-known and has multiple forms. We adopt one common definition. [8, Eqs. (1) and (16)].

$$H_{\mathbf{j}}(\mathbf{z};\Sigma) := \frac{(-1)^{|\mathbf{j}|}}{\phi_{\mathbf{z}}(\mathbf{z};\Sigma)} \left(\frac{\partial}{\partial \mathbf{z}}\right)^{\mathbf{j}} \phi_{\mathbf{z}}(\mathbf{z};\Sigma) \tag{3}$$

$$\phi_{\mathbf{z}}(\mathbf{z};\Sigma) := (2\pi)^{-N/2} (\det \Sigma)^{-1/2} \exp\left(-\frac{1}{2}\mathbf{z}^T \Sigma^{-1} \mathbf{z}\right). \tag{4}$$

$$\mathbf{z} = (x, p_x, y, p_y, z, \delta).$$

The phase space has 6 dimensions, thus $N = 6$ and $\mathbf{j}$ has 6 components. The symbol $\mathbf{j} = (j_1, \ldots, j_6)$ denotes the degree of the multi-variable Hermite-Gaussian polynomial.

This basis is suitable for expanding the beam phase space distribution function ρ. Since the diffusion coefficient $\langle D_{ij}\rangle$ can be viewed as a functional of ρ, it can also be associated with Hermite-Gaussian polynomials.

We could represent our phase space distribution as a series of Hermite-Gaussian polynomials.

$$\rho(\mathbf{z}) = \sum_{\mathbf{n}\in\mathbb{N}^6} a_{\mathbf{n}} H_{\mathbf{n}}(\mathbf{z};\Sigma)\phi_{\mathbf{z}}(\mathbf{z};\Sigma). \tag{5}$$

Then the diffusion coefficient becomes:

$$\begin{aligned}\langle D_{ij}\rangle &= \frac{1}{2N}\iiint (q_iq_j)d^3\mathbf{x}d^3\mathbf{p}_1d^3\mathbf{p}_2\\ &\sum_{\substack{\mathbf{m}\in\mathbb{N}^6\\ \mathbf{n}\in\mathbb{N}^6}} a_{\mathbf{n}}H_{\mathbf{n}}(\mathbf{x},\mathbf{p}_1;\Sigma)\phi_{\mathbf{z}}(\mathbf{x},\mathbf{p}_1;\Sigma)a_{\mathbf{m}}H_{\mathbf{m}}(\mathbf{x},\mathbf{p}_2;\Sigma)\phi_{\mathbf{z}}(\mathbf{x},\mathbf{p}_2;\Sigma)\\ &\int\frac{md^3\mathbf{p}_1'}{E_1'}\int\frac{md^3\mathbf{p}_2'}{E_2'}|M_c|^2\frac{m^2}{E_1E_2}\frac{\delta^4(p_1'^{\mu}+p_2'^{\mu}-p_1^{\mu}-p_2^{\mu})}{(2\pi)^2}.\end{aligned}\tag{6}$$

From simple observation, the diffusion coefficient consists of many similar components, namely $\langle D_{ij}^{mn}\rangle$.

$$\begin{aligned}\langle D_{ij}^{\mathbf{mn}}\rangle &= \frac{1}{2N}\iiint (q_iq_j)d^3\mathbf{x}d^3\mathbf{p}_1d^3\mathbf{p}_2\\ &H_{\mathbf{n}}(\mathbf{x},\mathbf{p}_1;\Sigma)\phi_{\mathbf{z}}(\mathbf{x},\mathbf{p}_1;\Sigma)H_{\mathbf{m}}(\mathbf{x},\mathbf{p}_2;\Sigma)\phi_{\mathbf{z}}(\mathbf{x},\mathbf{p}_2;\Sigma)\\ &\int\frac{md^3\mathbf{p}_1'}{E_1'}\int\frac{md^3\mathbf{p}_2'}{E_2'}|M_c|^2\frac{m^2}{E_1E_2}\frac{\delta^4(p_1'^{\mu}+p_2'^{\mu}-p_1^{\mu}-p_2^{\mu})}{(2\pi)^2}.\end{aligned}\tag{7}$$

Consequently,

$$\langle D_{ij}\rangle = \sum_{\mathbf{m}\in\mathbb{N}^6}\sum_{\mathbf{n}\in\mathbb{N}^6} a_na_m\langle D_{ij}^{\mathbf{mn}}\rangle.\tag{8}$$

To facilitate understanding, we introduce the generating function:

$$\sum_{\mathbf{j}\in\mathbb{N}^6}\frac{\mathbf{t}^{\mathbf{j}}}{\mathbf{j}!}H_{\mathbf{j}}(\mathbf{z};\Sigma) = \exp\left(\mathbf{t}^{\top}\Sigma^{-1}\mathbf{z}-\frac{1}{2}\mathbf{t}^{\top}\Sigma^{-1}\mathbf{t}\right).\tag{9}$$

Combining the definition of Gaussian distribution in Eq. (4) gives.

$$\sum_{\mathbf{n}\in\mathbb{N}^6}\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!}H_{\mathbf{n}}(\mathbf{z};\Sigma)\phi_{\mathbf{z}}(\mathbf{z};\Sigma) = \exp\left(\frac{1}{2}\left(\mathbf{z}-\mathbf{t}\right)^{\top}\Sigma^{-1}\left(\mathbf{z}-\mathbf{t}\right)\right).\tag{10}$$

In order to calculate $\langle D_{ij}^{\mathbf{mn}}\rangle$, we have to calculate the auxiliary function G_{ij} defined with generating functions.

$$\begin{aligned}G_{ij}(\mathbf{t},\mathbf{s}) &:= \frac{1}{2N}\iiint (q_iq_j)d^3\mathbf{x}d^3\mathbf{p}_1d^3\mathbf{p}_2\\ &\sum_{\mathbf{n}\in\mathbb{N}^6}\frac{\mathbf{t}^{\mathbf{n}}}{\mathbf{n}!}H_{\mathbf{n}}(\mathbf{x},\mathbf{p}_1;\Sigma)\phi_{\mathbf{z}}(\mathbf{x},\mathbf{p}_1;\Sigma)\\ &\sum_{\mathbf{m}\in\mathbb{N}^6}\frac{\mathbf{s}^{\mathbf{m}}}{\mathbf{m}!}H_{\mathbf{m}}(\mathbf{x},\mathbf{p}_2;\Sigma)\phi_{\mathbf{z}}(\mathbf{x},\mathbf{p}_2;\Sigma)\\ &\int\frac{md^3\mathbf{p}_1'}{E_1'}\int\frac{md^3\mathbf{p}_2'}{E_2'}\\ &|M_c|^2\frac{m^2}{E_1E_2}\frac{\delta^4(p_1'^{\mu}+p_2'^{\mu}-p_1^{\mu}-p_2^{\mu})}{(2\pi)^2}.\end{aligned}\tag{11}$$

Substituting the generating function Eq. (10) into Eq. (11) yields.

$$\begin{aligned}G_{ij}(\mathbf{t},\mathbf{s}) &= \frac{1}{2N}\iiint (q_iq_j)d^3\mathbf{x}d^3\mathbf{p}_1d^3\mathbf{p}_2\\ &\exp\left(\frac{1}{2}\left(\mathbf{z_1}-\mathbf{t}\right)^{\top}\Sigma^{-1}\left(\mathbf{z_1}-\mathbf{t}\right)\right)\\ &\exp\left(\frac{1}{2}\left(\mathbf{z_2}-\mathbf{s}\right)^{\top}\Sigma^{-1}\left(\mathbf{z_2}-\mathbf{s}\right)\right)\\ &\int\frac{md^3\mathbf{p}_1'}{E_1'}\int\frac{md^3\mathbf{p}_2'}{E_2'}|M_c|^2\frac{m^2}{E_1E_2}\frac{\delta^4(p_1'^{\mu}+p_2'^{\mu}-p_1^{\mu}-p_2^{\mu})}{(2\pi)^2},\end{aligned}\tag{12}$$

where $\mathbf{z}_1 = (\mathbf{x},\mathbf{p}_1)$, $\mathbf{z}_2 = (\mathbf{x},\mathbf{p}_2)$.

If we can calculate $G_{ij}(\mathbf{t},\mathbf{s})$ analytically (Eq. (12)), then we can perform the multi variate Taylor expansion of $G_{ij}(\mathbf{t},\mathbf{s})$.

$$G_{ij}(\mathbf{t},\mathbf{s}) = \sum_{\mathbf{m}\in\mathbb{N}^6}\sum_{\mathbf{n}\in\mathbb{N}^6}\frac{\partial^{\mathbf{n}}}{\mathbf{n}!}\frac{\partial^{\mathbf{m}}}{\mathbf{m}!}G_{ij}(\mathbf{t},\mathbf{s})|_{\mathbf{t}=\mathbf{s}=0}\mathbf{t}^{\mathbf{n}}\mathbf{s}^{\mathbf{m}}.\tag{13}$$

Comparing Eq. (7), Eq. (11) and Eq. (13), we obtain the final result.

$$\langle D_{ij}^{\mathbf{mn}}\rangle = \partial^{\mathbf{n}}\partial^{\mathbf{m}}G_{ij}(\mathbf{t},\mathbf{s})|_{\mathbf{t}=\mathbf{s}=0}.\tag{14}$$

The above discussion implies that the key problem in evaluating the simplified diffusion coefficient is to accurately compute Eq. (12).

We can simplify Eq. (12) via variable substitution:

$$\begin{cases}\mathbf{z}_1^{\mathbf{new}} = \mathbf{z}_1-\frac{\mathbf{s}+\mathbf{t}}{2}\\ \mathbf{z}_2^{\mathbf{new}} = \mathbf{z}_2-\frac{\mathbf{s}+\mathbf{t}}{2}\end{cases}.\tag{15}$$

Then Eq. (12) becomes

$$\begin{aligned}G_{ij}(\mathbf{t},\mathbf{s}) &= \frac{1}{2N}\iiint (q_iq_j)d^3\mathbf{x}d^3\mathbf{p}_1d^3\mathbf{p}_2\\ &\exp\left(\frac{1}{2}\left(\mathbf{z}_{\mathbf{1}}^{\mathbf{new}}-\frac{\mathbf{t}-\mathbf{s}}{2}\right)^{\top}\Sigma^{-1}\left(\mathbf{z}_{\mathbf{1}}^{\mathbf{new}}-\frac{\mathbf{t}-\mathbf{s}}{2}\right)\right)\\ &\exp\left(\frac{1}{2}\left(\mathbf{z}_{\mathbf{2}}^{\mathbf{new}}-\frac{\mathbf{s}-\mathbf{t}}{2}\right)^{\top}\Sigma^{-1}\left(\mathbf{z}_{\mathbf{2}}^{\mathbf{new}}-\frac{\mathbf{s}-\mathbf{t}}{2}\right)\right)\\ &\int\frac{md^3\mathbf{p}_1'}{E_1'}\int\frac{md^3\mathbf{p}_2'}{E_2'}|M_c|^2\frac{m^2}{E_1E_2}\frac{\delta^4(p_1'^{\mu}+p_2'^{\mu}-p_1^{\mu}-p_2^{\mu})}{(2\pi)^2}.\end{aligned}\tag{16}$$

In order to keep the expression simple, we have omitted the supperscript "new" on $\mathbf{x},\mathbf{p},\mathbf{x}',\mathbf{p}'$. We remind the readers that these quantities are the components of new variables $\mathbf{z}_1^{\mathbf{new}},\mathbf{z}_2^{\mathbf{new}},\mathbf{z}_1'^{\mathbf{new}},\mathbf{z}_2'^{\mathbf{new}}$.

According to our unpublished work [7], the double-Gaussian diffusion coefficient can be simplified as:

$$G_{ij}(\mathbf{t},\mathbf{s}) = \frac{\pi\alpha^2mN}{\Gamma}L_c\int\frac{d^3\Delta}{|\Delta|}e^{-\tilde{S}}\left(\delta_{ij}-\frac{\Delta_i\Delta_j}{|\Delta|^2}\right)\tag{17}$$

$$\begin{aligned}\tilde{S} = \sum_{a,b=1}^{3} &A_{ab}\left(\Delta_a-\frac{p_{0a}}{2}\right)\left(\Delta_b-\frac{p_{0b}}{2}\right)\\ &-B_{ab}\left(\frac{z_{0b}}{2}\left(\Delta_a-\frac{p_{0a}}{2}\right)+\frac{z_{0a}}{2}\left(\Delta_b-\frac{p_{0b}}{2}\right)\right)\\ &+\frac{1}{4}C_{ab}z_{0a}z_{0b}.\end{aligned}$$

The coefficients A_{ij},B_{ij},C_{ij} come from the components of Σ^{-1}

$$-\frac{1}{2}\mathbf{z}^T\Sigma^{-1}\mathbf{z} = -\sum_{i,j=1}^{3}\frac{1}{2}A_{ij}p_ip_j+B_{ij}p_ix_j+\frac{1}{2}C_{ij}x_ix_j,$$

Γ is the normalization factor.

$$\Gamma = \int\exp\left(-\frac{1}{2}\mathbf{z}^T\Sigma^{-1}\mathbf{z}\right)d^6\mathbf{z},$$

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and z_{0a}, p_{0a} are the spatial and momentum components of the vector $\mathbf{s} - \mathbf{t}$.

The idea of simplifying Eq. (16) to Eq. (17) is similar to the simplifications in the Bjorken–Mtingwa model or the work of Kubo and Oide [9]. For brevity, we do not present the derivation of Eq. (17) here.

Compared to Eq. (16), Eq. (17) involves only three integration variables. Consequently, this integration can potentially be evaluated numerically.

Application Procedure

Based on the above discussion, we propose the following procedure to evaluate the simplified diffusion coefficient $\langle D_{ij} \rangle$ for an arbitrary phase space distribution.

1. **Obtain the equilibrium distribution.** Use particle tracking or other methods to acquire the equilibrium phase space distribution ρ, which is determined by both linear and nonlinear beam dynamics.
2. **Expand ρ into Hermite–Gaussian series.** At each position s along the storage ring, expand ρ as a series of Hermite–Gaussian polynomials Eq. (5). The coefficients $a_{\mathbf{n}}$ can be obtained using a generalized Fourier expansion method.
3. **Compute the auxiliary function and its derivatives.** Numerically evaluate the auxiliary function G_{ij} using Eq. (17). Then, compute the partial derivatives of G_{ij} at the origin to obtain the basic components $\langle D_{ij}^{\mathbf{mn}} \rangle$ via Eq. (14). This step is also performed numerically.
4. **Sum all components.** Finally, substitute the coefficients and components into Eq. (8) to obtain the total simplified diffusion coefficient $\langle D_{ij} \rangle$.

CONCLUSION

This paper presents a generalized analytical approach for calculating the simplified diffusion coefficient of IBS, applicable to arbitrary beam phase space distributions. For advanced accelerator-based light sources pursuing high brightness and coherence, smaller emittance and shorter bunch lengths make IBS a crucial factor for the final performance. Additionally, novel concepts such as SSMB lead to distinct non-Gaussian distributions, where traditional analytical models are inadequate and Monte Carlo methods are time-consuming with limited theoretical insight.

To adapt to such non-Gaussian cases, we first extend the IBS description from the traditional growth rate τ to the diffusion coefficient D_{ij} in the FPE. This extension have been mentioned in previous researches [6].We adopt Hermite-Gaussian polynomials as the basis for expanding arbitrary phase space distribution functions, and decompose the diffusion coefficient into components corresponding to Hermite-Gaussian terms of different degrees. We then employ the generating function method to reduce the calculation of the diffusion coefficient components for arbitrary distributions to the calculation of an auxiliary function. Through variable substitution, we simplify the auxiliary function and transform the high-dimensional integration problem into a numerically solvable one. Finally, A four-step practical procedure is also proposed for applying this method.

It should be noted that this paper still assumes IBS to be a slow and weak process, where phase space distributions are dominated by single-particle dynamics and the diffusion coefficient can be averaged over the entire phase space. It is not applicable to strong IBS regimes where the diffusion coefficient becomes a local phase space function and the full FPE must be solved without simplification.

Despite this limitation, the proposed approach enables more accurate IBS calculations for non-Gaussian distributions and bridges the gap between traditional analytical models and numerical methods in non-Gaussian IBS analysis. This work reveals how fine structures inside a bunch influences the IBS effect. It lays a theoretical foundation for intensional design beam structures to mitigate adverse IBS impacts. This work opens up a gate for the active control of the inherently random IBS process, offering an analytical tool for the development of next-generation high-brightness accelerator-based light sources and advanced storage ring concepts.

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