REPRESENTATION OF MECHANICAL MODES SPECTRUM

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Abstract

The static and dynamic Lorentz force detuning (LFD) derive from the sum of many mechanical modes and their interplay with EM radiation pressure. The LFD may give rise to instabilities in which the EM resonance frequency moves away from the "set point" value. The usual analytic theory for the oscillatory instability, and its threshold, are for a single, isolated mechanical mode; not a sum. We want a mathematical representation of the LFD spectrum that is compatible with an "isolated resonance" style of stability analysis, but nevertheless captures the phase-advance properties of a sum of many modes.

INTRODUCTION

We wish to emphasize four aspects of the mechanical modes spectra responsible for ponderomotive instability encountered in superconducting radio frequency (SRF) cavity resonators.

1. The absence of super-harmonic generation.
2. The collective behaviour of mechanical modes (MMs).
3. The crucial importance of mechanical breathing modes.
4. The representation of many interacting mechanical resonances. Provided that isolation criteria are satisfied, each mode can be replaced by a single resonance plus a complex number that is the collective effect of the other modes. Let K_k be the kth mode coupling.

Absence of RF Harmonics

The best known, and probably most studied, nonlinear resonator is the driven Duffing oscillator [1] encountered in mechanical systems; it has a cubic nonlinearity of the restoring force versus amplitude that introduces a shift of the resonance frequency. If this frequency falls with amplitude of oscillation, the system is called a softening resonator. An important feature in the response of the classical cubic oscillator is the appearance of harmonics of the drive frequency. The cubic term triples the drive frequency. Sum and difference frequencies between the fundamental and third harmonic introduce second and fourth harmonic, and so on.

In contrast to that behaviour, the SRF resonator (with a cubic dependence of resonance frequency on amplitude due to the Lorentz-force detuning) does not display super-harmonic response. The nonlinearity enters through the agency of mechanical modes, whose spectrum extends into the kHz but not into MHz or GHz. The slow response of these modes results in averaging of very many RF cycles, and so there is no frequency multiplication of the RF fundamental. Thus the SRF resonator behaves as a driven mean-square oscillator [2]. This prediction is not widely known; and the absence of RF harmonics needs to be confirmed by measurement.

Ponderomotive Instability

The Lorentz-force detuning (LFD) may give rise to a so-called ponderomotive instability in which the EM resonance frequency moves away from the desired or "set point" value. There is an analytic theory of this instability: due to Schulze [3] for the generator driven case, and to Delayen [4] for the self-excited resonator. Usually it is stated that there are two types of instability: a monotonic variation that occurs near DC; and an oscillatory variation that occurs in the neighbourhood of a strong mechanical mode (one having large quality factor and coupling constant). Schulze treats these individually: a sum of modes at DC for the monotonic, and a single mode for the oscillatory. Delayen uses the DC and AC response of the same, single mode for both types of instability. In essence, these theories for the oscillatory instability, and its threshold, are for a single, isolated mechanical mode. Of course, there may be several such strong modes. This raises the question of how to represent the effect of many interacting mechanical modes.

COLLECTIVE BEHAVIOUR OF MODES

The RF cavity resonator is supposed to have some idealised shape, which is considered a thin shell. Any deformations of that shape are described by a superposition of mechanical eigenmodes each of which has a shape function and a damped sinusoidal time dependence. In principle, there is an infinite set of these modes. The cavity shape in the steady state is a superposition of modes, each excited at zero frequency. For example, an external mechanical tuner that deforms the cavity relies on a superposition of modes.

The mechanical modes always act collectively. At DC, the modes cooperate to produce exactly the steady state deformation of the shell. If the cavity shell is excited mechanically at "low frequency", the modes act coherently to follow the time-dependent deformation. The response of a single mode when excited at its resonance frequency is Q_m times larger than the DC response of that mode. If the modes have large mechanical quality factors Q_m, then "low frequency" means below the first mechanical resonance. If the modes have small quality factors, then "low frequency" can extend well above the first resonance. When the quality factors are large, we may think of the response (of the modes) as being described by either of two simplified models: (i) when the excitation frequency is between the individual resonances, the collective response is coherent; and (ii) when the excitation frequency lies within the bandwidth of a single high-Q mode, the response ceases to be collective and is almost entirely the response of the individual mode j if $K_j Q_j \gg \sum_k K_k$. In this second model, the residual response of all the other modes contributes a very small amplitude and phase shift. This way of thinking about the collective response and its mathematical description is elaborated in Ref. [5].

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ROLE OF SYMMETRY

If the cavity has axial symmetry, then the mechanical modes can be separated into two classes: longitudinal and transverse. Examples of transverse modes are breathing and bending modes; these modes are respectively symmetric and anti-symmetric about the cavity axis. Slater's theorem [6] makes the link between mechanical deformation of the cavity shape and the shift of the EM resonance frequency. Spatial symmetry determines [7] which mechanical modes couple to changes of the EM resonance frequency. The EM field shape of the accelerating TM mode is symmetric about the cavity axis. If the deformation shape has the same spatial symmetry (i.e. overlaps) as the EM field shape, then the coupling is strong. The longitudinal mechanical modes couple quite strongly to the resonance frequency of the EM eigenmode used for particle acceleration. If the deformation shape has alternating equal and opposite zones (e.g. is anti-symmetric), then the overlap integral with the EM field shape is zero. Hence, transverse bending modes (alone) cannot couple to the EM resonance frequency.There are, however, second order mechanical effects which yield a quadratic dependence of frequency shift on mode displacement; but the effect is very small and results in frequency doubling.

Symmetry is a strong argument. How then, do transverse vibrations (i.e. microphonics) or a radial piston couple to the EM resonance frequency? The resolution of this question introduces three key considerations: (i) asymmetric deformation, (ii) combination of different transverse mode types, and (iii) coherence of mechanical modes.

The radial piston deforms one side of the azimuthal cavity wall, leading to an asymmetric shape. The breathing modes are symmetric, and the bending modes are antisymmetric across the diameter. Hence the piston couples to the sum of breathing and bending modes, which together yield an asymmetric deformation. At low frequency, all mechanical modes of one type act collectively; and the total deformation is the superposition of both mode types, bending and breathing. So, at low frequency, the cavity shape follows the deformation imposed by the piston. Any microphonic deformation of the cavity shell that is described by a superposition of breathing modes alone or in combination with other transverse mode types will also couple to and may alter the EM eigenfrequency, see Fig. 1.

APPLICABILITY OF AN "ISOLATED RESONANCE" THEORY

Whereas the phase change across a truly single resonance is π, the phase change is zero to π (or less) back to zero across each resonance within a series. This does not necessarily invalidate a theory constructed for a single resonance. An instability may occur when the amplitude response is large; this is the region a few bandwidths wide about the resonance frequency. The bandwidth is ω_0/Q and $Q \gg 1$. In this region the phase change (below to above resonance) is (roughly) zero to π. It is only outside and above this region that the phase response reverts to zero; and this is where the

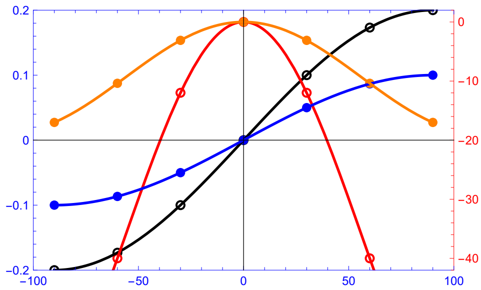

Figure 1: Displacement (blue, black) and EM frequency shift (orange, red) versus mechanical oscillation phase for 1st bending mode of TRIUMF 9-cell cavity. Left ordinate = displacement (mm). Right ordinate = frequency shift (Hz). Abscissa = MM oscillation phase (deg.) Open and filled circles are data points. Solid curves are sinusoidal fits to data. Calculations courtesy of Yan-Yun Ma (TRIUMF) 2018.

amplitude response is tiny. A resonance in a series (ordered by natural frequency) may be considered isolated (acting like a pure single resonance) when two conditions are met: neighbour resonances are (i) separated by many bandwidths; and (ii) the resonance strengths are similar. Contrastingly, if one resonance strength strongly dominates, and encroaches within a few bandwidths of the other, it will influence the phase response of its weaker neighbour – and the "isolated resonator" assumption is invalid.

PIECEWISE REPRESENTATION OF MANY RESONANCES

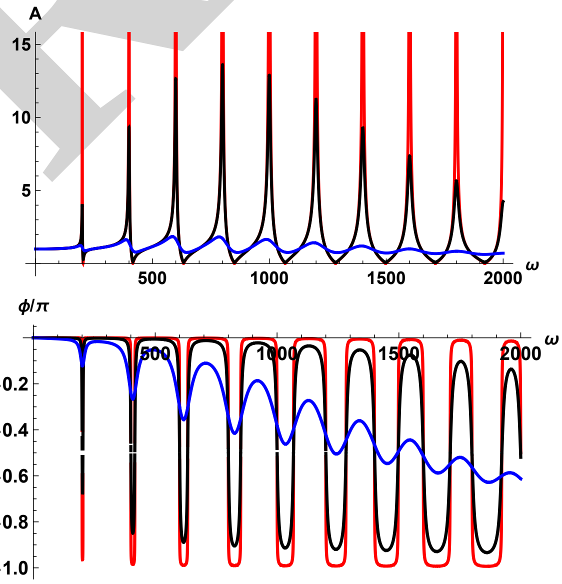

Figure 2: Sum of Resonators Spectrum. Upper: amplitude versus frequency. Lower plot: phase versus frequency. Blue curves: $Q_M = 10$. Black curves: $Q_M = 10^2$. Red curves: $Q_M = 10^3$. The amplitude response of the latter (red curve) has been truncated, and really extends up to $A = 135$.

We propose that the longitudinal (or transverse) mechanical mode (MM) spectrum can be approximated by a slowly varying "background function" (we label it H) plus one or more localised resonances. Previously Luong [8] and Neumann [9] modeled the background by a low-pass filter. Their approach is model-dependent, and there is no strong physics motivation for the low-pass filter. Further, their two-way partition and series cut-off are artificial. We prefer a representation that is model independent, that neither adds nor subtracts physics information, and that can be derived from theory or data (measured spectra).

Let K_k be the coupling constant (from displacement to EM frequency shift), and Q_k and Ω_k be the quality factor and resonance frequency of the kth mechanical mode. The transfer function between frequency shift and mechanical displacement is the sum over modes:

$$H(s) = \sum_k h_k(s), \quad h_k(s) \equiv \frac{K_k \Omega_k^2}{s^2 + (\Omega_k/Q_k)s + \Omega_k^2} \,. \qquad (1)$$

All modes (label k) contribute to every Laplace frequency s.

In the neighbourhood of a particular mode, the background function is the sum over all other resonators (MMs) but excluding the particular resonance itself. If the number of "other" resonators is large, then H will be smooth. The only assumption made is that the number of modes is large, a few dozen or more, so the background function is smoothly varying. It is a simple and elegant representation: in the neighbourhood of mode k, the response is modeled by:

$$\begin{aligned} H(s) &= h_k(s) + H_k(s) \quad \text{where} && (2)\\ H_k(s) &\equiv \sum_{j\neq k} h_j(s) \approx \sum_{j\neq k} h_j(i\Omega_k) = R_k + iX_k \,. && (3)\end{aligned}$$

The sum runs over $j = 1, 2, 3 \ldots \infty$. This is a piecewise definition of $H(s)$, with a different H_k for each resonator. Thus the local behaviour of the kth mode is approximated by:

$$H(s) \approx h_k(s) \;+\; R(\Omega_k) + iX(\Omega_k) \,. \qquad (4)$$

At low frequency $R \gg |X|$, so X may be omitted. If the adjustable parameters are obtained by fitting to data, then this is performed piecewise. One way to subdivide the frequency range is from one amplitude minimum to the next.

The slowly varying background impedance $H_k(s)$ can be constructed from the measured LFD spectrum of the dressed cavity. Thus $h_k + H_k$ is found piecewise by fitting of resonator parameters (centre frequency Ω_k, quality factor Q_k, coupling constant K_k) and background parameters R_k and X_k to the measured LFD frequency response in the neighbourhood of the local resonance. Locally the response is $H(i\omega) = h_k + H_k$. For the fitting to experimental data, the in-phase and quadrature components of the LFD response correspond to the real and imaginary parts of the sum $h_k + H_k$ (where each is complex). Suppose S_I are samples of the in-phase, and S_Q are samples of the quadrature response at the sample (drive) frequencies. Fitting implies to select $\Omega_k, Q_k, K_k, R_k, X_k$ consistent with both data sets; and trade-offs may be required, leading to manual intervention. E.g. if the resonance is very broad or exceedingly narrow, automated fitting software may fail.

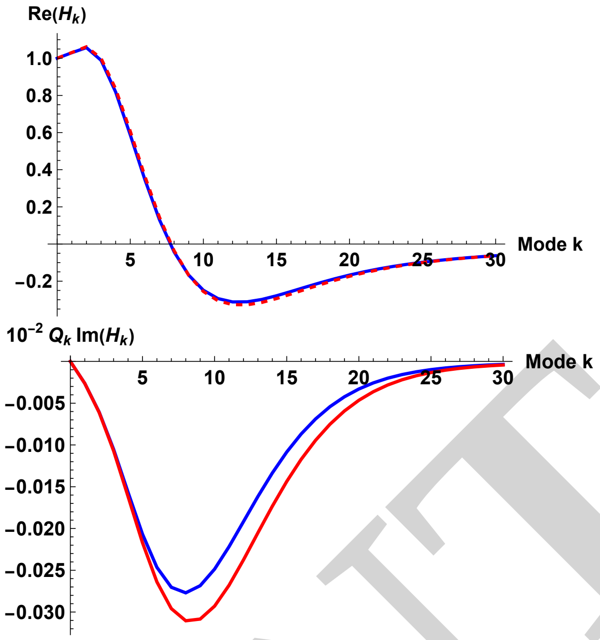

Figure 3: Real and Imaginary parts of H_k versus frequency of mode number k. Upper plot: $\Re[H_k]$. Blue curve is $Q_M = 10$; red-dashed curve is $Q_M = 10^3$. Lower: $\Im[H_k] \times (Q_M/100)$. Blue curve is $Q_M = 10$; red curve $Q_M = 10^3$.

Example

We take an artificial spectrum of individual modes: the mode frequency spacing and quality factors are equal, and the coupling constants are $K_k = k^2 e^{-k/2}/S$ with $S = \sum_{k=1}^{\infty} k^2 e^{-k/2}$. Figure 2 shows the spectrum of the sum of modes for three values of Q_M. In particular notice the phase plot for the low-Q resonators, which interact so strongly that the nett phase of the ensemble does not reach $-\pi$ over the range of the plot.

We form the samples $H_k(i\Omega_k)$ and plot the real and imaginary parts $R_k = \Re[H_k]$ and $X_k = \Im[H_k]$. It transpires that the shape and amplitude range of the real part is insensitive to Q_M. Contrastingly, the amplitude range of the imaginary part scales (almost) as $1/Q_M$, while the shape is insensitive to Q_M. Figure 3 Upper is an overlay of R_k for $Q = 10$ and $Q = 10^3$; the curves are almost indistinguishable. Figure 3 Lower is an overlay of $X_k \times (Q/10^2)$ for $Q = 10$ and $Q = 10^3$; the curves differ by only a few percent. If we had compared $X_k \times Q$ for $Q = 10^2$ and $Q = 10^3$, we would find them almost indistinguishable.

CONCLUSION

We have presented several insights concerning the mechanical mode spectra responsible for ponderomotive instability. We propose a simple piecewise definition of the transfer function of summed mechanical resonators. Each piece is the sum of a particular isolated resonance plus a complex number whose value represents the effect of all other resonators. This innovation facilitates stability analysis in the case that the MMs do not behave as isolated resonances.

REFERENCES

[1] *The Duffing Equation: Nonlinear Oscillators and their Behaviour*, I. Kovačic and M. J. Brennan, Eds. Chichester, UK: John Wiley & Sons, 2011. doi:10.1002/9780470977859

[2] S. Koscielniak, "Mean-Square Resonator – and Relation to the Duffing Resonator", TRIUMF, Vancouver, BC, Canada, Rep. TRI-BN-24-36, Nov. 2024.

[3] D. Schulze, "Ponderomotive Stability of RF Resonators and Resonator Control Systems", Kernforschungszentrum Karlsruhe, Karlsruhe, Germany, Rep. KFK-1493, 1971; translation: Argonne National Laboratory, Lemont, IL, USA, Rep. ANL-TRANS-944, 1972. doi:10.2172/4659728

[4] J. R. Delayen, "Phase and amplitude stabilization of superconducting resonators", D.Phil. thesis, California Institute of Technology, Pasadena, CA, USA, 1978.

[5] S. Koscielniak, "Lorentz force detuning – mechanical resonance representation", TRIUMF, Vancouver, BC, Canada, Rep. TRI-BN-25-23R, Dec. 2025.

[6] J. C. Slater, "Microwave electronics", *Rev. Mod. Phys.*, vol. 18, no. 4, pp. 441–512, Oct. 1946. doi:10.1103/RevModPhys.18.441

[7] M. Doleans and S.-H. Kim, "Insights in the physics of the dynamic detuning in SRF cavities and its active compensation", in *Proc. Particle Accelerator Conf. (PAC'03)*, Portland, OR, USA, 2003, paper TPPE024, pp. 1599–1601.

[8] M. Luong *et al.*, "Analysis of microphonic disturbances and simulation for feedback compensation", in *Proc. Eur. Particle Accelerator Conf. (EPAC'06)*, Edinburgh, Scotland, 2006, paper THPCH159, pp. 3167–3169.

[9] A. Neumann *et al.*, "Microphonics in C.W. TESLA cavities and their compensation with fast tuners", in *Proc. 13th Int. Workshop on RF Superconductivity (SRF'07)*, Beijing, China, 2007, paper WE201, pp. 377–383.