ANALYTICAL SOLENOID MATCHING ROUTINES WITH COOLPY[*] [†]

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Abstract

Muon colliders require strong beam cooling to reduce the large phase space of muon beams produced from pion decay. The final stage of ionization cooling employs high-field solenoids, absorbers, and RF cavities, where precise beam matching is essential to avoid emittance growth. In this work, we present a beam-parameter–based approach to design and optimize solenoid lattices for the final cooling channel. The method models realistic solenoid fields and solves the coupled beam envelope equation while accounting for momentum changes in absorbers and RF systems. To implement this approach efficiently, we developed the Python package coolpy, which computes the evolution of Twiss parameters and optimizes matching coil settings. Two case studies demonstrate matched beam transport in solenoid-based beamlines.

INTRODUCTION

The concept of a muon collider was first proposed in the late 1960s [1] and has the potential to represent a paradigm shift in accelerator and particle physics [2]. One of the main challenges arises from the short lifetime of the muon, 2.2 µs, in its rest frame. As a result, substantial effort has been invested in creating new accelerator technologies that can generate intense muon beams. These technologies must quickly accelerate the muons to TeV-scale energies before a large portion of them decay. A muon collider would provide a powerful probe for physics beyond the Standard Model (SM). It would simultaneously enable high-precision studies within the SM itself, like the investigations of the Higgs self coupling [3] and of lepton-parton distribution functions [4].

Muon Ionization Cooling

This paper focuses on the technological developments required for a muon collider, in particular muon ionization cooling [5]. Muon ionization cooling is a rapid method used to reduce the beam emittance of muons that were originally produced in proton-target interactions.

The resulting muon beam initially spans a large phase space, which must be substantially compressed to reach the high luminosities required for a collider. Ionization cooling decreases the emittance of a muon beam by passing the muons through an energy-absorbing medium, where their momentum is reduced by ionization losses. The longitudinal momentum is subsequently restored using radio-frequency (RF) acceleration. Repetition of these stages multiple times can progressively reduce the phase space of the muon beam.

In this paper, we discuss the final cooling channel [6, 7], which is the last part of the ionization cooling complex [8] of the proposed muon collider. The final cooling channel consists of a sequence of low- and high-field transport solenoids, absorbers, and RF cavity systems. Its purpose is to reduce the transverse emittance while keeping the growth of longitudinal emittance under control. The principal challenge is to optimize this final cooling system so that the optical beam parameters are correctly matched. If this condition is not met, an increase in emittance will arise and must be carefully prevented.

When designing such a final cooling system, four main aspects must be considered:

- a precise representation of the solenoid's magnetic field, taking its fringe regions into account,
- the dynamical evolution of the muon beam momentum in the absorber and RF cavities,
- a proper description of the beam parameters in a transversely coupled system, and
- an appropriate optimization procedure.

Coolpy: A Solenoid Based Lattice Design Solution

In principle, it is possible to optimize such an ionization cooling system using macro-particle tracking codes. However, these methods are computationally expensive. A faster approach is to solve the beam parameters for one or a few reference particles and tune the solenoid settings once specific targets are met. This multi-reference particle optimization is especially useful for bunches with large relative momentum spread, as in ionization cooling. After an optimized configuration is obtained, the same solenoid setup can then be applied in full multi-particle tracking simulations.

We have developed a code that follows this beam-parameter–based optimization approach and named it coolpy, which we introduce here for the first time. In the following, we provide a brief discussion of the evolution of beam parameters within a solenoid and the corresponding description of the solenoidal magnetic field. We next present two beam case studies using coolpy and conclude with a summary.

BEAM PHYSICS IN SOLENOIDS

Transverse Coupled Motions

The uncoupled motion of single or multiple charged particles in accelerators is described using the Courant–Snyder [9] or Twiss [10] formalism. In the presence of a solenoid, a

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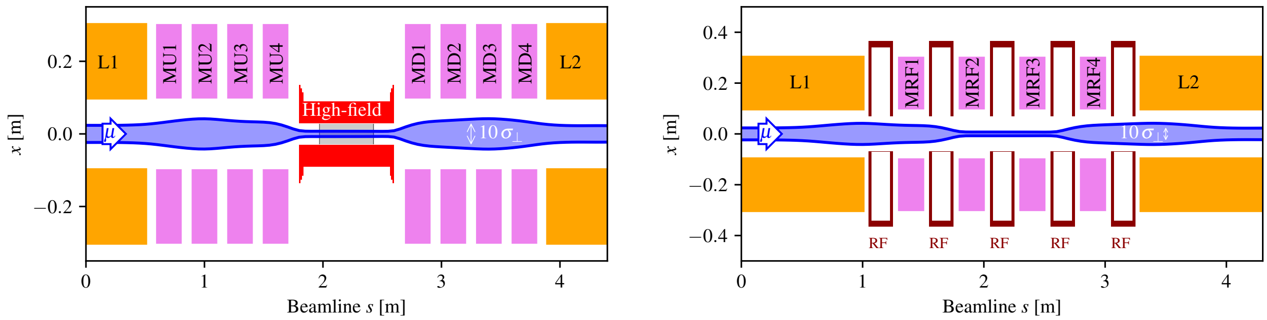

Figure 1: Left: Solenoid beam line with an absorber along with the associated beam aperture that directs the beam from a 4 T low-field region into a 40 T high-field region and back. Right: Arrangement of RF cavities separated by spaces that allow the installation of the matching coils in between.

charged particle beam is simultaneously focused in both the horizontal (x) and vertical (y) directions. This introduces a coupling between these planes. Consequently, the conventional Twiss parameterization needs to be modified to incorporate this coupled dynamics through the use of the canonical angular momentum

$$\langle L_{\mathrm{can}}\rangle = \langle x p_y - y p_x\rangle + \frac{0.3B}{2}\left(\langle x^2\rangle + \langle y^2\rangle\right). \tag{1}$$

In Eq. (1), x and y denote transverse spatial coordinates, while p_x and p_y correspond to the transverse momenta. The focusing strength appearing in. The brackets $\langle\cdot\rangle$ represent the average taken over an ensemble of particles. The Twiss parameters in solenoids were introduced by G. Penn et al. [11] and their definitions in the cartesian frame are

$$\beta_\perp = \frac{\langle x^2\rangle + \langle y^2\rangle}{2m_\mu c\,\varepsilon_\perp^{\mathrm{N}}}, \quad \alpha_\perp = -\frac{\langle x p_x\rangle + \langle y p_y\rangle}{2m_\mu c\,\varepsilon_\perp^{\mathrm{N}}}, \tag{2}$$

which was taken by reference [12]. In Eq. (2), the symbol $\varepsilon_\perp^{\mathrm{N}}$ denotes the four-dimensional, normalized transverse emittance and m_μ the muon mass. The evolution of the Twiss parameters along a beamline consisting of solenoids is governed by the beam envelope equation

$$2\beta_\perp\beta_\perp'' - (\beta_\perp')^2 + 4\beta_\perp^2\kappa^2 - 4(1+\mathcal{L}^2) = 0, \tag{3}$$

where $\beta_\perp' = -2\alpha_\perp$ and $\mathcal{L} = \langle L_{\mathrm{can}}\rangle/2\varepsilon_\perp$ denotes the normalized canonical angular momentum. Eq. (1) is given by $\kappa \approx 0.15B[\mathrm{T}]/p_z[\mathrm{GeV}/c]$. Since both p_z and B vary along the beamline in an ionization cooling lattice, this variation must be included in κ when solving Eq. (3).

Field Description in Solenoids

E. Callaghan and S. Maslen proposed an efficient analytical method to model realistic solenoid fields [13]. In their framework, a set of coaxial wire loops is represented by an infinitesimally thin cylindrically symmetric current sheet. The magnetic field generated by such a current sheet is then characterized by two formulas: one for the radial component, B_ϱ, and one for the axial component, B_z. The magnetic field of a finite solenoid can then be obtained by superposing the fields generated by multiple coaxial current sheets.

LATTICE DESIGN CASE STUDIES

Accurate matching conditions are especially important in an ionization cooling system because the bunch has relatively low beam energies. In the center of the solenoid, the transverse beam correlation parameter must obey $\beta_\perp'' = 0$ and $\beta_\perp' = 0$. Accordingly, the matched condition for a solenoidal channel can be written as

$$\beta_\perp = \frac{1}{\kappa}\sqrt{1+\mathcal{L}^2}. \tag{4}$$

A charged particle beam traveling from one solenoid to the next must satisfy the condition given in Eq. (4) in order to remain matched. To fulfill this requirement, additional matching coils are placed between these solenoids. For a single particle, two independent coil parameters are sufficient to achieve the matching. In muon cooling, the bunch typically exhibits a relatively large momentum spread, which can be described using several reference particles. In general, when the beam is modeled using N independent reference particles, a total of $2N$ free coil parameters is needed. In this work, the muon bunch is represented by two reference particles, which is typically sufficient in practice. Their momenta are chosen around the mean bunch momentum p_0, with offsets corresponding to the momentum spread, i.e., $p_0 \pm \sigma_p$. In the following case studies, four geometrically identical matching coils are used. The matching is achieved by independently adjusting the current density in each coil.

In ionization cooling, the momentum of charged particles changes due to absorbers and RF systems. These dynamical momentum changes within a beamline must be taken into account when solving Eq. (3). In this work, two different ionization cooling configurations are presented as illustrative cases. The first example considers the passage of a positive muon bunch through a liquid hydrogen absorber installed inside an ultra-high-field solenoid [14, 15], which is being developed at CERN. The second example examines a beamline composed of solenoids and RF pillbox cavities.

Muon Beam Passage Through an Absorber

The general decrease in muon momenta within an absorber, together with a schematic representation of the associated aperture, is illustrated on the left-hand side of Fig. 1.

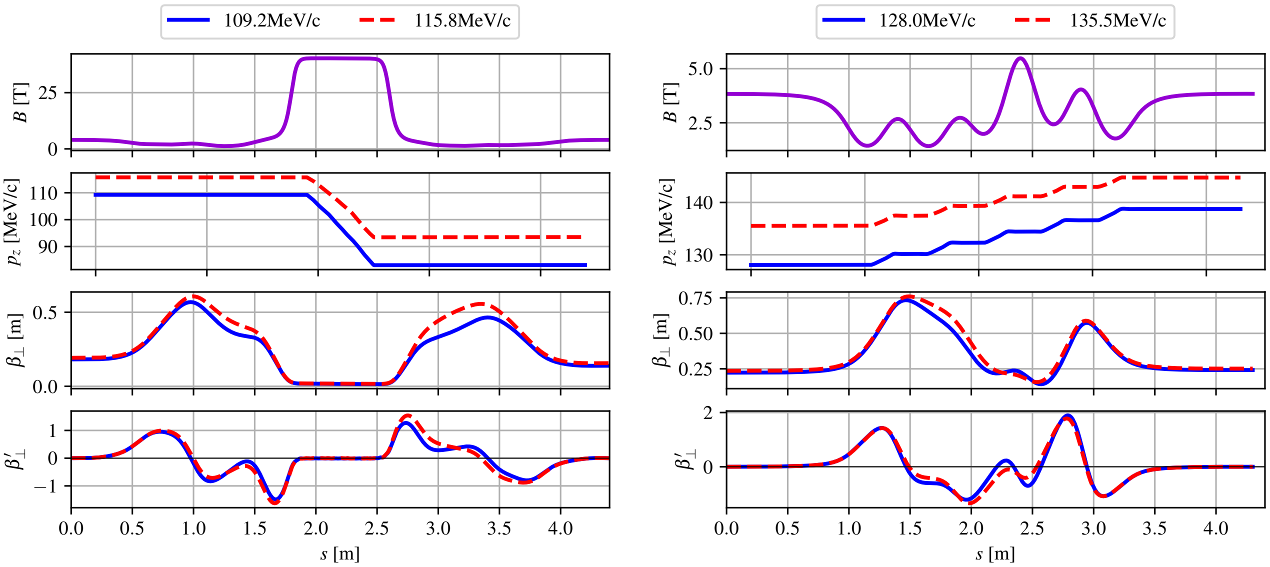

Figure 2: Case studies of the dynamical matching procedure. The solenoid beamline with an absorber is illustrated on the left and the lattice including RF cavities is depicted on the right side. The plots show the on-axis magnetic field B_z, the momentum evolution of the reference particles, and the optimized Twiss parameters $\beta_\perp$ and $\beta'_\perp$ obtained from the matching method.

To maximize the reduction in normalized transverse emittance in the absorber, a hydrogen absorber is placed inside a CERN-type 40 T solenoid. The goal is to guide the beam from a 4 T low-field region into a high-field region and back, while avoiding emittance growth from mismatches. The muon bunch analyzed in this work has a mean reference momentum of $p_{\mathrm{ref}} = 112.5\,\mathrm{MeV/c}$ and a momentum spread of $\sigma_{p_z} = 3.3\,\mathrm{MeV/c}$. As shown in the left panel of Fig. 2, the muon momenta are reduced in the absorber section as a result of ionization energy losses. To fulfill the matching condition defined in Eq. (4), eight matching coils are installed in this beamline configuration—four upstream and four downstream—as shown on the left side of Fig. 1. The current densities of the matching coils were tuned. The Twiss parameters of the reference particles after optimization are shown on the left in Fig. 2.

RF Acceleration in Solenoid-Based Systems:

An ionization cooling channel also incorporates RF cavities to restore the energy lost by muons in the absorber. Because the RF cavities are separated by finite gaps, these spaces provide convenient locations for matching coils, as shown schematically on the right of Fig. 1. The upper-right plots of Fig. 2 show the longitudinal magnetic field on-axis B_z and the corresponding momentum evolution. The lower-right plots show the matched parameters for the two reference particles after optimizing the coil currents.

COOLPY

We implemented the solenoid field model, the numerical solution of the envelope equation Eq. (3), and the optimization procedure in the Python package coolpy. Multiple solenoid elements can be assembled into a beamline, fully including overlapping fringe fields between adjacent components, with magnetic field components B_z and B_ϱ available at any spatial point. For a specified beamline configuration, coolpy determines the single-particle Twiss parameters $\beta_\perp$ and $\beta'_\perp$ by numerically solving Eq. (3) using a fourth-order Runge–Kutta scheme [16–18]. It performs this autonomously while accounting for the evolution of the momentum $p_z(s)$. The optimization of the coil settings for several reference particles is carried out with the Nelder–Mead algorithm [19]. A step-by-step tutorial with coolpy examples is available on the GitHub page [20].

CONCLUSION

A beam-parameter–based method was presented to design and optimize solenoid lattices for muon ionization cooling channels, modeling realistic solenoid fields and solving the coupled beam envelope equation with varying momentum and magnetic fields. It determines the matched beam conditions using multiple reference particles to represent the large momentum spread in ionization cooling. To efficiently perform this optimization, the Python package coolpy was developed, which computes the Twiss parameter evolution and optimizes matching coil settings. Two case studies demonstrate how the method enables matched beam transport in solenoid-based cooling channels. The optimized coil configurations obtained with coolpy are intended to serve as starting points for subsequent full multi-particle tracking simulations, where stochastic effects such as multiple scattering and energy straggling can be included; step-by-step examples are provided on the project's GitHub page [20].

REFERENCES

[1] F. F. Tikhonin, “On the effects at colliding mu meson beams”, JINR, Dubna, Russia, Rep. JINR-P2-4120, May 2008. doi:10.48550/arXiv.0805.3961

[2] C. Accettura *et al.*, “Towards a muon collider”, *Eur. Phys. J. C*, vol. 83, 2023. doi:10.1140/epjc/s10052-023-11889-x

[3] P. Andreetto *et al.*, “Aspects of Higgs physics at a $\sqrt{s} = 3$ TeV Muon Collider with detailed detector simulation”, *Eur. Phys. J. C*, vol. 85, no. 3, p. 221, Mar. 2025. doi:10.1140/epjc/s10052-025-13923-6

[4] R. Capdevilla, F. Garosi, D. Marzocca, and B. Stechauner, “Testing the neutrino content of the muon at muon colliders”. *J. High Energy Phys.*, vol. 2025, no. 168, Apr. 2025. doi:10.1007/JHEP04(2025)168

[5] D. Neuffer, “Principles and applications of muon cooling”, in *Proc. 12th Int. Conf. on High Energy Accelerators*, F. T. Cole and R. Donaldson, Eds., Batavia, IL, USA, Aug. 1983, pp. 481–484, Rep. LA-UR-83-2375.

[6] R. B. Palmer, R. C. Fernow, and J. L. Lederman, “Muon Collider Final Cooling in 30-50 T Solenoids”, in *Proc. PAC’11*, New York, NY, USA, Mar.-Apr. 2011, paper THOBN2, pp. 2061–2063.

[7] H. Sayed, R. B. Palmer, and D. Neuffer, “High field – low energy muon ionization cooling channel”, *Phys. Rev. Spec. Top. Accel. Beams*, vol. 18, no. 9, p. 091001, Sep. 2015. doi:10.1103/PhysRevSTAB.18.091001

[8] C. Rogers, “Preliminary report on key WP4 subsystems”, MuCol Collaboration, EU Project 101094300, Rep. v1, Nov. 2025. doi:10.5281/zenodo.17733000

[9] E. D. Courant and H. S. Snyder, “Theory of the alternating-gradient synchrotron”, *Ann. Phys.*, vol. 3, no. 1, pp. 1–48, Jan. 1958. doi:10.1016/0003-4916(58)90012-5

[10] R. Q. Twiss and N. H. Frank, “Orbital stability in a proton synchrotron”, *Rev. Sci. Instrum.*, vol. 20, no. 1, pp. 1–17, Jan. 1949. doi:10.1063/1.1741343

[11] G. Penn and J. S. Wurtele, “Beam envelope equations for cooling of muons in solenoid fields”, *Phys. Rev. Lett.*, vol. 85, no. 4, pp. 764–767, Jul. 2000. doi:10.1103/PhysRevLett.85.764

[12] R. C. Fernow, “Physics analysis performed by ECALC9”, Fermi National Accelerator Laboratory, Batavia, IL, USA, Rep. FERMILAB-MUCOOL-280, Sep. 2003.

[13] E. E. Callaghan and S. H. Maslen, “The magnetic field of a finite solenoid”, NASA Lewis Research Center, Cleveland, OH, USA, Rep. NASA-TN-D-465, Oct. 1960. https://ntrs.nasa.gov/citations/19980227402

[14] C. Accettura *et al.*, “Mechanical design of a ReBCO non/metal-insulated 40 T solenoid for the Muon Collider”, *IEEE Trans. Appl. Supercond.*, vol. 35, no. 5, pp. 1–5, Aug. 2025. doi:10.1109/TASC.2024.3504472

[15] R. Unterrainer, “Advances in UHF solenoids for HEP”, presented at Workshop on Ultra-High-Field Solenoids, CERN, Geneva, Switzerland, Nov. 25–26, 2025. https://indico.cern.ch/event/1572785/contributions/6722889/

[16] C. Runge, “Ueber die numerische Auflösung von Differentialgleichungen”, *Math. Ann.*, vol. 46, no. 2, pp. 167–178, Jun. 1895. doi:10.1007/BF01446807

[17] W. Kutta, “Beitrag zur naeherungsweisen Integration totaler Differentialgleichungen”, *Z. Angew. Math. Phys., 46, 435–453*, 1901.

[18] W. H. Press, S. A. Teukolsky, W. T. Vetterling, and B. P. Flannery, *Numerical Recipes: The Art of Scientific Computing*, 3rd ed. Cambridge, UK: Cambridge University Press, 2007.

[19] J. A. Nelder and R. Mead, “A simplex method for function minimization”, *Comput. J.*, vol. 7, no. 4, pp. 308–313, Jan. 1965. doi:10.1093/comjnl/7.4.308

[20] Coolpy examples, Github repository, 2026. https://github.com/BerndStechauner/coolpy/tree/main/Examples