EFFECT OF BROKEN SUPERPERIODICITY ON HALF INTEGER RESONANCE AND POSSIBLE COMPENSATION*

H. Bartosik and F. Zimmermann†, CERN, Geneva, Switzerland
G. Franchetti‡, GSI, Darmstadt, HFHF and Goethe Universität, Frankfurt, Germany

Abstract

With a perfect two-fold superperiodicity in a storage ring, the half integer resonance is not excited and can be overlapped by the beam tune spread induced, e.g., by beam-beam collisions in a collider, space charge in a hadron storage ring, or energy spread and chromaticity, without detrimental effect on the beam. If the superperiodicity of the quadrupolar lattice is broken, either by design or due to machine imperfections, a resonance stopband is created, which may lead to unstable particle motion. Guided by historical literature, analytical calculations, and toy simulations, we explore the relation between single-particle motion and optics near the resonance. We discuss possible mitigation approaches and the construction of tuning knobs or modified lattices which can reduce the resonance strength. Our discussion here focuses on the linear transverse optics and extends to chromatic resonance effects and linearized space charge forces.

INTRODUCTION

In a storage ring, the equation for the transverse motion of a single particle satisfies the linear Hill's equation [1], also known as the Ermakov-Pinney equation [2–4]:

$$y'' + k(s)y = 0, \, , \tag{1}$$

with $'$ denoting the derivative with respect to the distance along the reference path s, and k a periodic focusing function, $k(s) = k(s + C)$, with C equal the ring circumference.

Considering a pattern of quadrupole focusing errors (Δk), in second order perturbation theory, Courant and Snyder derived the width of the stopband near a half integer resonance $Q \approx n/2$, due to these errors, as [1]

$$\delta Q_n \approx \frac{1}{2\pi} \left| \int_0^C \beta(s)\, \Delta k(s)\, e^{i2\mu(s)}\, ds \right|, \tag{2}$$

and the maximum beta beating as [1]

$$\left(\frac{\Delta\beta}{\beta}\right)_{\max} \approx \frac{|\delta Q_n|}{2|Q - n/2|}\,, \tag{3}$$

where μ is the betatron phase.

In general, however, it is not obvious how to distinguish quadrupole errors $\Delta K = \int \Delta k ds$ from the design values of the integrated quadrupole strengths K, especially if the design focusing structure is not (super)periodic. In addition, the second-order approximations in Eqs. (2) and (3) should break down close to the half-integer resonance.

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† frank.zimmermann@cern.ch
‡ g.franchetti@gsi.de

SINGLE GRADIENT ERRORS

Consider a single gradient error of integrated strength ΔK at $s = 0$. According to Eq. (2) the width of the stopband is

$$\delta Q_n \approx \frac{1}{2\pi}\, \beta(0)\, \Delta K\,. \tag{4}$$

For example, with $\Delta K = 0.01\ \mathrm{m}^{-2}$ and $\beta(0) = 10$ m, the stop band is expected to be $\delta Q_n \approx 0.1/(2\pi) \approx 0.0159$.

Consider the 2×2 transport matrix

$$\mathbf{M}(\mu) = \begin{pmatrix} \cos\mu & \beta \sin\mu \\ -(1/\beta)\sin\mu & \cos\mu \end{pmatrix} \tag{5}$$

and add a thin-lens quadrupole with integrated strength K described by

$$\mathbf{M}_\mathrm{q}(K) = \begin{pmatrix} 1 & 0 \\ -K & 1 \end{pmatrix}, \tag{6}$$

so that the product matrix becomes $\mathbf{M}_\mathrm{tot} = \mathbf{M}(\mu) \cdot \mathbf{M}_\mathrm{q}(K)$. The tune Q of $\mathbf{M}_\mathrm{tot}$ can be obtained as

$$Q = \mathrm{Im}\left[\frac{\mathrm{arccosh}\,(\mathrm{Tr}(\mathbf{M}_\mathrm{tot})/2)}{2\pi}\right] \tag{7}$$

and, in case of unstable motion, the exponential increment per turn, λ, in units of 2π, by the real part of the square brackets of Eq. (7). The dependence of Q and λ on the base tune $\mu/(2\pi)$ is illustrated in Fig. 1. The stop band is clearly visible, characterized by a constant tune Q and nonzero growth rate λ. The total extent of the stopband is close to the expected width $\delta Q_n \approx 0.016$ from Eq. (4).

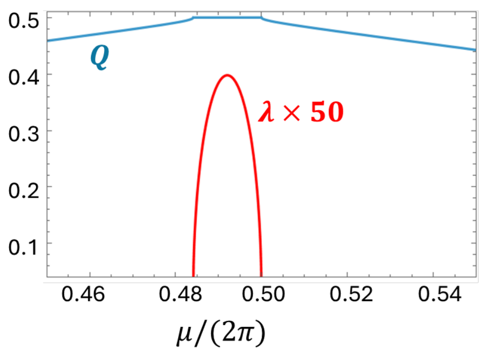

Figure 1: Tune Q and growth rate λ (amplified by a factor 50) as a function of the base tune $\mu/(2\pi)$, with $\beta = 10$ m, and for a single additional quadrupole of strength $K = 0.01\ \mathrm{m}^{-1}$.

STOPBAND COMPENSATION

Now we add a second quadrupole of equal strength after half a period, *i.e.*, we consider $\mathbf{M}_{\text{tot},2} = \mathbf{M}(\mu/2) \cdot \mathbf{M}_{\text{q}}(K) \cdot \mathbf{M}(\mu/2) \cdot \mathbf{M}_{\text{q}}(K)$. The second quadrupole leads to a two-fold superperiocity, which makes the linear half-integer stopband disappear (cf. Fig. 2). Instead of restoring the superperiodicity, we could also install a second quadrupole of negative strength $-K$ a betatron phase advance of π away from the first one, so that we obtain $\mathbf{M}_{\text{tot},3} = \mathbf{M}(\mu-\pi) \cdot \mathbf{M}_{\text{q}}(-K) \cdot \mathbf{M}(\pi) \cdot \mathbf{M}_{\text{q}}(K)$. This solution also compensates the stopband and the overall tune shift, as shown in Fig. 3.

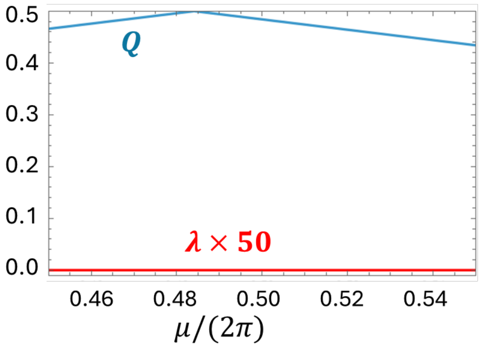

Figure 2: Tune Q and growth rate λ (amplified by a factor 50) as a function of the base tune $\mu/(2\pi)$, with $\beta = 10$ m, and for two additional quadrupoles of strength $K = 0.01\ \text{m}^{-1}$, half a period apart, as in the matrix $\mathbf{M}_{\text{tot},2}$.

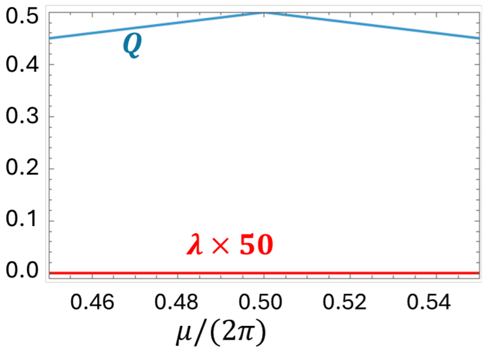

Figure 3: Tune Q and growth rate λ (amplified by a factor 50) as a function of the base tune $\mu/(2\pi)$, with $\beta = 10$ m, and for two additional quadrupoles of strength $\pm K = 0.01\ \text{m}^{-1}$, a betatron phase advance of π apart, as in the matrix $\mathbf{M}_{\text{tot},3}$.

For a generic storage ring, according to Eq. (2) two corrector quadrupole magnets, of strengths $\Delta K_{1,\text{c}}$ and $\Delta K_{2,\text{c}}$, placed a phase advance $\Delta\mu = \pi/4$ apart should suffice to fully compensate the stopband in one plane, by requiring

$$\beta_1\,\Delta K_{1,\text{c}} e^{i2\mu_1} + \beta_2\,\Delta K_{2,\text{c}} e^{i2(\mu_1+\pi/4)} = -\frac{1}{2\pi}\int_0^C \beta(s)\,\Delta k(s)\,e^{i2\mu(s)}\,ds\,, \tag{8}$$

where μ_1 denotes the betatron phase at the first corrector quadrupole. Considering both transverse planes, generalizing the above, a set of 4 quadrupoles is required to either compensate the 1/2 integer stopband in x and y, or to compensate it in one plane only, while not affecting the other.

From the discussion here, it might appear as if the superperiodicity is not important for compensating the half-integer resonance. However, later in this paper, we will illustrate that a (restored) superperiodicity preserves the stopband compensation also in the presence of additional lattice-dependent perturbations like chromaticity or (linear) space charge, which most of the solutions of Eq. (8) do not.

MAINTAINING SUPERPERIODICITY

Considering a synchrotron consisting of N identical periods, the one-turn-map reads $\mathbf{M} = \mathbf{M}_1^N$ with $\mathbf{M}_1$ the transport map of one period. The stability of the motion is determined by the eigenvalues λ of $\mathbf{M}$. If λ is imaginary the motion is stable, and unstable if λ is real. This condition is equivalently expressed in terms of the trace of the one turn map $|\,\text{Tr}\,\mathbf{M}|$, namely the particle dynamics is stable for $|\,\text{Tr}\,\mathbf{M}| < 2$. If the lattice is composed of N equal periods, i.e., perfectly superperiodic, then $|\,\text{Tr}\,\mathbf{M}| = |\lambda_1^N + 1/\lambda_1^N|$, where λ_1 denotes the eigenvalue of $\mathbf{M}_1$.

Evidently, in case λ_1 is imaginary, λ_1^N will be imaginary, too. Therefore, it is expected that a superperiodic structure is stable if its periods are. The requirement of stability is, here, expressed through transport maps of the periods.

Noteworthily, it is possible that one of the periods (say the period j) internally *differs* from the others, but that its optics is fully matched to the symmetric optics at the entrance and the exit of period j: If β, α and the phase advance $\Delta\phi$ are made to be the same as for all the other periods, then the transport map of this period $\mathbf{M}_j$ will have exactly the same components as all the others, and the stability of motion will be unchanged. This means that $|\text{Tr}\mathbf{M}|$ will be unchanged. Hence, if the superperiodic structure was stable on the half integer resonance, it will also remain stable with a broken superperiodicity for the modified fully matched case.

This no longer holds true if the phase advance per cell is not fully matched, which is the case, e.g., if the phase advance $\Delta\phi$ of period j differs from those in the other periods (although β, α may still be matched). In this case $\mathbf{M}_j = \mathbf{M}_1 + \delta\mathbf{M}_j$, and the one turn map reads $\mathbf{M}_B = \mathbf{M} + \mathbf{M}_a\delta\mathbf{M}_j\mathbf{M}_b$, with $\mathbf{M} = \mathbf{M}_a\mathbf{M}_j\mathbf{M}_b$. Therefore the trace reads $\text{Tr}\,\mathbf{M}_B = 2\cos(2\pi Q) + \text{Tr}\big(\mathbf{M}_a\delta\mathbf{M}_j\mathbf{M}_b\big)$.

CHROMATICITY & SPACE CHARGE

We now consider linear perturbations whose strength depends on the optical functions. An example is the chromaticity experienced by off-momentum particles, another the linear space-charge force affecting the particle motion in the beam center. Here, even if one period is modified but fully matched, the linear perturbation arising from chromaticity or space charge breaks the condition of superperiodicity. Hence, the resulting map sequence includes a perturbation matrix. This mechanism will create a stopband of instability in case the tune of the non-superperiodic lattice is set too

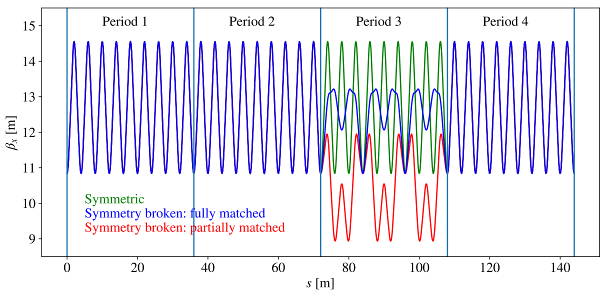

Figure 4: Beta function for three lattices: a strictly superperiodic lattice (green); one modified period, but with full optics matching (blue); only partially matched period (red).

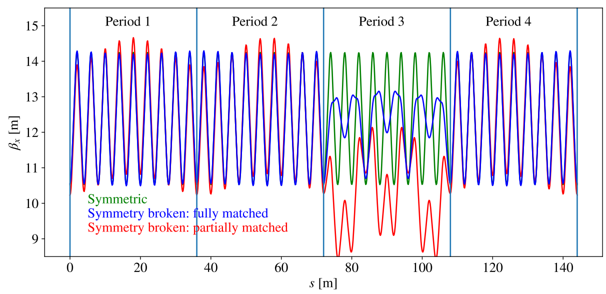

Figure 5: Beta function around the ring at $\delta p/p = -2.5 \times 10^{-2}$, for the same three lattices as in Fig. 4. The blue line in the periods 1,2,4 is different than the green.

close to the half integer, even if the modified period is linearly matched to the neighboring periods at both its entrance and exit, and if it has the same total betatron phase advance as all the other periods.

The breaking of the superperiodicity induced by chromaticity will be a function of $\Delta p/p$, and, accordingly, each particle will encounter a different stopband width and instability growth rate, depending on its momentum offset.

Figure 4 presents an illustrating example. The green curve shows a regular β_x function for a sequence of identical FODO cells. Four (super)periods are delimited by the vertical lines. The maps for each superperiod are equal, and all yield the same trace $\mathrm{Tr}\,\mathbf{M}_1 = \mathrm{Tr}\,\mathbf{M}_2 = \mathrm{Tr}\,\mathbf{M}_3 = \mathrm{Tr}\,\mathbf{M}_4 = -1.9308$, with $\beta_x \approx 10.837$ m, and $\alpha_x \approx 0$ at the entrance and exit of each period. The blue curve exhibits an optics modified in the 3$^{\mathrm{rd}}$ period, which, in this case, is constructed so as to be (nearly) matched both in β_x and α_x, and also in the phase advance $\Delta\phi$. As a result, the linear matrix $\mathbf{M}_3$ becomes almost equal to the map of all other periods, namely $\mathrm{Tr}\,\mathbf{M}_3 \approx -1.9308$, although the optics inside the period is different. Finally, the red curve shows the case where the optics of the third period is matched in β_x, and α_x, but not in $\Delta\phi$. In this case, the trace of $\mathbf{M}_3$ differs from those of $\mathbf{M}_1 = \mathbf{M}_2 = \mathbf{M}_4$, namely $\mathrm{Tr}\,\mathbf{M}_3 \approx -1.8949$. The imperfect matching for the last case (red) renders $\mathbf{M}_3$ different from the maps of the other periods, which for tunes near the half integer leads to a one-turn matrix $\mathbf{M}$ with $|\mathrm{Tr}\,\mathbf{M}| > 2$, and, hence, generates an instability.

If we repeat the same procedure, but now consider a particle with $\delta p/p = -2.5 \times 10^{-2}$ we find that $\mathrm{Tr}\,\mathbf{M}_1 = \mathrm{Tr}\,\mathbf{M}_2 = \mathrm{Tr}\,\mathbf{M}_4 = -1.9642$, which expresses the superperiodic change of tunes due to the chromaticity. For the superperiodic case (green) $\mathrm{Tr}\,\mathbf{M}_3 = -1.9642$, while for the partially matched 3rd period (red) $\mathrm{Tr}\,\mathbf{M}_3 = -1.8548$, which — not surprisingly — indicates a transport map different from $\mathbf{M}_1$. The most interesting case is the one where the 3rd period is fully matched, but with an internal optics different from the other periods. In this case the presence of chromaticity yields $\mathrm{Tr}\,\mathbf{M}_3 = -1.9574$, which demonstrates that the third period, due to chromaticity, produces an off-momentum "partially matched optics", where the off-momentum particle motion becomes unstable on the half integer. Figure 5 shows the beta function for an off-momentum particle considering the same three lattices as in Fig. 4.

CHROMATIC COMPENSATION

The situation encountered in the last section cannot be mitigated by a conventional resonance correction strategy where compensating elements are set so as to induce driving terms of opposite sign. A more effective strategy is consistently modifying another period of the accelerator in order to create a globally compensating effect $M'_a \delta M_{\mathrm{cor}} M'_b$, such that $\mathrm{Tr}\,\mathbf{M}_B(\Delta p/p) = 2\cos(2\pi Q^*) + \mathrm{Tr}(\mathbf{M}_a\ \delta\mathbf{M}(\Delta p/p)\ \mathbf{M}_b) + \mathrm{Tr}\big(\mathbf{M}'_a\ \delta\mathbf{M}_{\mathrm{cor}}(\Delta p/p)\ \mathbf{M}'_b\big) \overset{!}{\approx} 2\cos(2\pi Q^*)$, with $Q^* \equiv Q(\Delta p/p)$. This approach, based on tailoring the beta functions in specific sub-periods of the lattice, is completely different from the standard chromatic correction using sextupole magnets. A similar procedure could restore the superperiodicity in presence of linear space charge.

CONCLUSIONS

A perfect two-fold superperiodicity in a storage ring efficiently suppresses the half-integer resonance. In the presence of quadrupole errors, the half-integer resonance can be compensated by employing quadrupole correctors, or by matching the perturbed lattice period with the unperturbed optics on either end. However, linear optics perturbations, e.g., due to chromaticity or space charge, whose strength depend on the optical functions, break the superperiodicity. A possible compensation strategy could then be to suitably modify another optical period in order to create a robust globally compensating effect. At the Future Circular Collider (FCC) [5], the technical insertions serve for different purposes (e.g. injection, collimation, ...), requiring special optics functions. For the FCC-ee "Global Hybrid Correction" optics [6], a universal insertion [5, p. 1.2.3] maintained the superperiodicity. The approach followed for the newer "Local Chromatic Correction" lattice [7] is to construct the straight sections with transparency conditions to preserve periodicity for on- and off-momentum particles including at large transverse amplitudes (c.f. Ref. [8]).

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