THE IMPACT OF TRANSVERSE-LONGITUDINAL COUPLING ON LONGITUDINAL MICROWAVE INSTABILITY

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Abstract

In a storage ring with an extremely small global phase slippage, the bunch length can vary significantly within one turn due to the partial phase slippage and transverse-longitudinal coupling, which means the adiabatic approximation usually adopted for longitudinal dynamics breaks down. The impact of such a bunch length variation and exchange of the head and tail part of the beam arising from the partial phase slippage on the coherent synchrotron radiation (CSR) induced longitudinal microwave instability (MWI) threshold has recently been theoretically investigated by some of the authors. In this work, we have extended the study to consider the influence of transverse-longitudinal coupling.

INTRODUCTION

Accelerator-based light source is an important direction in the current development of accelerator. At present, there are two main types of these light sources, storage ring-based synchrotron radiation light sources and linear accelerator-based free electron lasers (FELs). They can provide high repetition rate and high peak power radiation, respectively. A novel storage ring-based light source mechanism known as steady-state microbunching (SSMB) is under investigation [1-4], which is promising to combine the advantages of these two sources and generate fully coherent radiation with both high repetition rate and high peak power. Such a SSMB ring exhibits extremely broad prospects in both scientific research and industrial applications.

The classical approach to achieving microbunches in an electron storage ring is to design a quasi-isochronous magnetic lattice with a very small global phase slippage factor η, which is based on the "zero-current" bunch length scaling law $\sigma_z \propto \sqrt{|\eta|}$ given by Sands [5]. However, when the global phase slippage factor becomes extremely small, the scaling law breaks down due to the impact of the partial phase slippage and transverse-longitudinal coupling. A more accurate expression for the bunch length is $\sigma_z = \sqrt{\epsilon_z \beta_z + \epsilon_x H_x}$, where ϵ_z and ϵ_x denote the longitudinal and horizontal emittance, and H_x represents the horizontal chromatic function. Note that under such conditions, the bunch length can vary significantly around the ring and is no longer a constant. A systematic analysis for the single-particle dynamics in this regime can be performed using Chao's SLIM formalism [6].

Coherent synchrotron radiation (CSR) induced longitudinal microwave instability (MWI) is the dominant collective effect limiting the beam current in electron storage rings with short bunches. Bane *et al.* have proposed a threshold formula applicable to conventional storage rings with the assumption that the bunch length remains constant around the ring within one turn [7]. Some of the authors extend the formula to account for the impact of the partial phase slippage, calculating the bunch wake based on the evolution of the bunch in longitudinal phase space over one revolution [8]. However, a more comprehensive analysis needs to consider the influence of transverse-longitudinal coupling. In this work, we therefore generalize the instability study to 4D phase space building on our earlier results. We find that a new dynamical characteristic emerges where particles with different betatron actions no longer behave identically when transverse-longitudinal coupling is taken into account. And we also propose a method for evaluating the threshold of CSR induced longitudinal MWI in this scenario. Although this work focuses on the longitudinal MWI, our results provide useful references for studies of other collective instabilities when there exists coupling between different dimensions.

THEORETICAL STUDY

To focus on the issues we concerned, here we only consider the horizontal and longitudinal dimensions and use the state vector $\mathbf{X} = (x, x', z, \delta)^T$. Under the assumptions that the storage ring is planar x-y uncoupled and the radio frequency (RF) cavities are placed at dispersion-free locations, which is the typical setup for present synchrotron light sources, the longitudinal bunch length can be expressed as $\sigma_z = \sqrt{\epsilon_z \beta_z + \epsilon_x H_x}$. The first term under the square root arises purely from longitudinal dimension, while the second term originates from transverse-longitudinal coupling. In conventional storage rings, the second term is typically much smaller than the first and can be neglected. Meanwhile, since the influence of the partial phase slippage is also insignificant, β_z can be approximated as a constant around the ring, leading to a fixed bunch length. However, when the global phase slippage factor is extremely small, the partial phase slippage factor causes β_z to vary drastically over one turn, and the contribution from transverse-longitudinal coupling is no longer negligible. These two effects together result in a strong variation of the bunch length along the entire ring, which is the scenario considered in this work.

In our previous work [8], we have investigated in detail the impact of the partial phase slippage on CSR induced

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longitudinal MWI. The partial phase slippage can cause bunch stretching and head-tail exchange of some particles within one turn. Under conditions that account for the partial phase slippage, we need to track the evolution of the bunch in the longitudinal phase space around the ring. In this case, the longitudinal wake experienced by the bunch over one turn is not merely a function of longitudinal position coordinate z and is also subject to the relative energy deviation δ, and can be written as

$$W_p(z,\delta) = \int_{\text{dipole}} \frac{W(z-\tilde{\eta}(s)C_0\delta)}{2\pi\rho}\,ds \qquad (1)$$

and

$$W(z) = \frac{2^{5/6}\rho^{1/3}}{3^{7/3}\epsilon_0\sigma_z^{4/3}}\left[\frac{\sqrt{\pi}\,q\,{}_1F_1\left(\frac{7}{6},\frac{3}{2},-\frac{q^2}{2}\right)}{\sqrt{3}\,\Gamma\left(\frac{5}{3}\right)} - \frac{2^{5/6}\Gamma\left(\frac{2}{3}\right){}_1F_1\left(\frac{2}{3},\frac{1}{2},-\frac{q^2}{2}\right)}{\Gamma\left(\frac{7}{3}\right)}\right], \qquad (2)$$

where $\tilde{\eta}$ is the partial phase slippage factor, s is the path length along the orbit, C_0 is the circumference of the ring, ρ is the bending radius of the trajectory, and $W(z)$ denotes the one-turn longitudinal bunch wake for a Gaussian bunch in a conventional storage ring [9] and it depends only on longitudinal position coordinate z, in which ϵ_0 is the vacuum permittivity, $q = z/\sigma_z$, and ${}_1F_1$ is a confluent hypergeometric function of the first kind. Note that σ_z varies along the ring.

When transverse-longitudinal coupling is further taken into account, a direct extension suggests that the calculation of longitudinal bunch wake over one turn requires an evaluation of bunch evolution in the four-dimensional phase space. The corresponding wake therefore depends not only on longitudinal coordinates but also on horizontal coordinates, which can be written as

$$W_{TLC}(x,x',z,\delta) = \int_{\text{dipole}} \frac{W(z-\tilde{\eta}(s)C_0\delta + R_{51}(s)x + R_{52}(s)x')}{2\pi\rho}\,ds. \qquad (3)$$

Thus, the primary impact of transverse-longitudinal coupling is analogous to that of the partial phase slippage. It modulates the amplitude of the wake by altering the bunch length and makes the wake dependent on more dimensions. In addition, a novel dynamical feature is that particles with different transverse phase space coordinates experience distinct longitudinal wake over one turn. Owing to the dependence of the threshold of longitudinal MWI on the longitudinal wake and the presence of betatron oscillation, a reasonable treatment is to average the longitudinal wake of individual particles with respect to betatron action and calculate the corresponding thresholds separately.

For particles with each betatron action J_x, the MWI threshold is calculated by employing the method proposed in our previous work [8]. We first calculate the energy spread σ_δ and the minimum bunch length along the ring to derive the kick divergence

$$K = 1.545\frac{N r_e \rho^{1/3}}{\gamma\sigma_z^{4/3}} \qquad (4)$$

and threshold

$$N_{\text{th}} = \frac{\pi\nu_s\gamma\sigma_\delta\sigma_z^{4/3}}{r_e\rho^{1/3}}, \qquad (5)$$

where N is the number of particles per bunch, r_e is the classical electron radius, γ is the Lorentz factor and ν_s is the synchrotron tune, without considering the bunch length variation. Then, we determine the kick divergence $K_{TLC}(J_x)$ and the rotation angle $R(J_x)$ of one-turn longitudinal bunch wake in longitudinal phase space for particles with each betatron action. Finally, we get the threshold

$$N_{\text{th}}(J_x) = \left(1 - D(J_x)\right)\frac{K}{K_{TLC}(J_x)}N_{\text{th}}, \qquad (6)$$

where $D(J_x) = 0.89e^{-0.44R(J_x)} + 0.11e^{2.31R(J_x)} - 1$ is a fitting formula. Since particles with different betatron actions have different instability thresholds, we take the minimum value of these thresholds as the global threshold of the bunch, which is given by

$$\widetilde{N_{\text{th}}} = \min N_{\text{th}}(J_x). \qquad (7)$$

SIMULATION RESULTS AND DISCUSSIONS

In this section, we will present the simulation results of a real lattice of the Metrology Light Source (MLS) of the Physikalisch-Technische Bundesanstalt (PTB) in Berlin [10] and compare them with our theoretical analysis. The lattice consists of one RF cavity and eight dipole magnets and Fig.1 shows the evolution of bunch length around the ring. Other relevant parameters are given in Table 1. The global phase slippage factor of this lattice is quite small at 6.6×10^{-5} and the effect of the partial phase slippage is not significant as indicated by the blue dashed line in Fig.1. In contrast, the effect of transverse-longitudinal coupling is remarkable as shown by the blue solid line in Fig.1 and it can stretch the bunch length by a factor of 2 to 3.

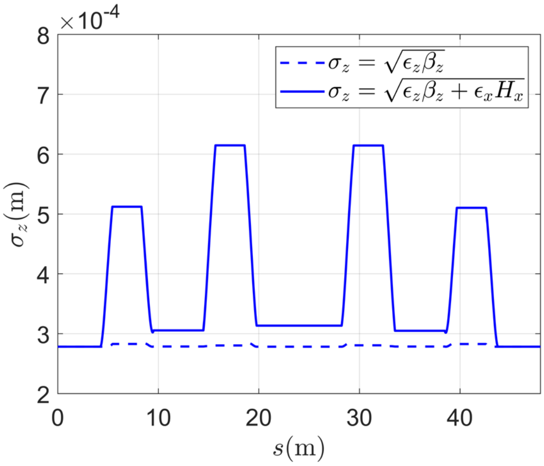

Figure 1: Bunch length evolution around the ring.

Table 1: Parameters used in the Simulation

Parameter	**Symbol**	**Value**
Ring circumference	C_0	48 m
Beam energy	E_0	630 MeV
RF voltage	V	480 kV
RF acceleration gradient	h	$0.008\ \mathrm{m}^{-1}$
Global phase slippage	η	6.6×10^{-5}
Bending radius	ρ	1.27 m
Bunch length	σ_z	2.8×10^{-4} m
Energy spread	σ_δ	4.4×10^{-4}
Longitudinal emittance	ϵ_z	123.8 nm
Horizontal emittance	ϵ_x	196.4 nm

Figure 2 presents the one-turn longitudinal bunch wake at horizontal actions of 0 and ϵ_x, which are calculated according to Eq. (3) and averaged, respectively. It can be seen that the wake varies with betatron action, consistent with our analysis. The wake at zero action has a larger amplitude than that at action of ϵ_x.

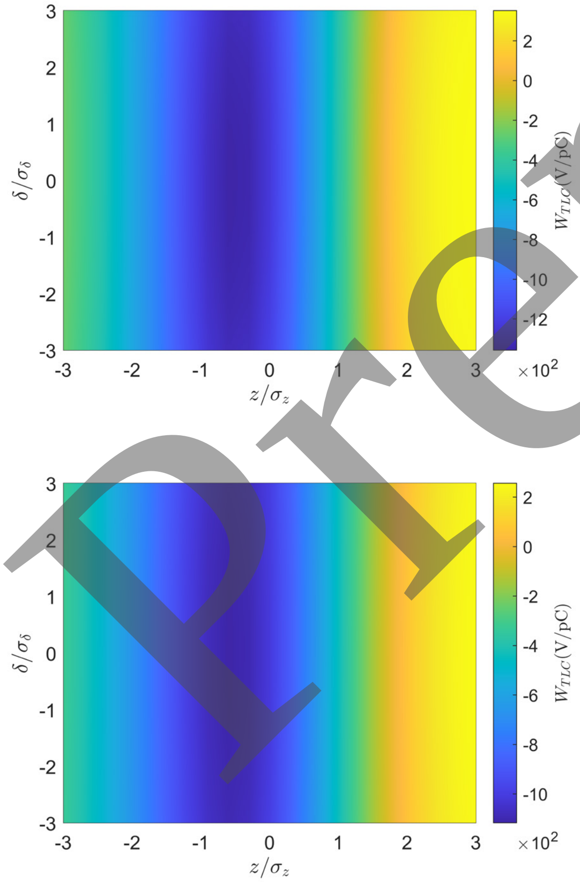

Figure 2: Longitudinal bunch wake at horizontal actions of 0 (top) and ϵ_x (bottom).

We calculate the thresholds corresponding to different betatron actions and take the minimum value as the global threshold of the bunch, which yields $\widetilde{N_{\mathrm{th}}} = 1.8 \times 10^7$. This value is about 2.3 times the threshold $N_{\mathrm{th}} = 7.7 \times 10^6$ obtained from classical formula without considering the variation of bunch length.

As particles with different betatron actions exhibit different instability thresholds, the conventional method commonly adopted in previous studies which takes a 5% increase in the overall beam energy spread relative to the natural energy spread as the criterion for instability onset is no longer applicable. In our simulations, we statistically investigate the variation of energy spread with the number of particles per bunch for particles in different betatron action ranges, and the results are shown in Fig. 3. It can be seen that the energy spread of particles with horizontal action lower than $0.5\epsilon_x$ start to show an evident growth at approximately $N = 2.0 \times 10^7$, while the energy spread of particles with horizontal action lower than $3.0\epsilon_x$ begins to rise at about $N = 4.0 \times 10^7$. This indicates that particles with different betatron actions behave differently and a fraction of them suffer instability in advance. $N = 2.0 \times 10^7$ is close to our calculated threshold, which demonstrates the satisfactory predictive capability of our newly proposed threshold evaluation method when considering transverse-longitudinal coupling.

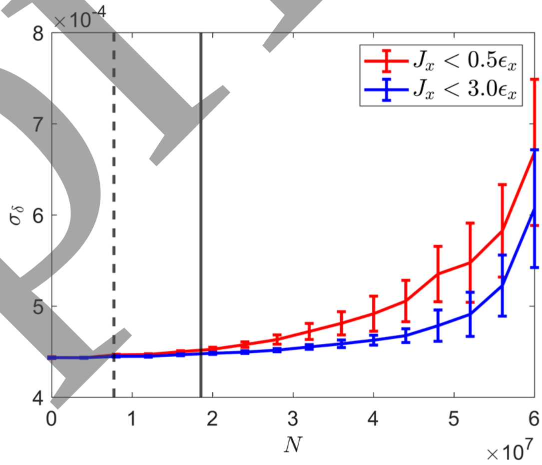

Figure 3: Energy spread as a function of bunch population for particles in different betatron action ranges. The black dashed line represents the classical prediction while the black solid line represents the threshold evaluated by our new method.

CONCLUSION

In this work, the impact of transverse-longitudinal coupling on CSR induced longitudinal MWI is investigated through theoretical analysis and numerical simulation. The key point is that the bunch can no longer be regarded as a single entity, since particles with different betatron actions exhibit distinct longitudinal dynamic behaviors. The presented analysis can also provide a reference for the study of other instabilities. Relevant experimental measurements will be carried out at MLS in future work.

REFERENCES

[1] D. F. Ratner and A. W. Chao, “Steady-state microbunching in a storage ring for generating coherent radiation”, *Phys. Rev. Lett.*, vol. 105, p. 154801, Oct. 2010. doi:10.1103/PhysRevLett.105.154801

[2] A. Chao *et al.*, “High power radiation sources using the steady-state microbunching mechanism”, in *Proc. IPAC'16*, Busan, Korea, May 2016, pp. 1048-1053. doi:10.18429/JACoW-IPAC2016-TUXB01

[3] X. J. Deng *et al.*, “Experimental demonstration of the mechanism of steady-state microbunching”, *Nature*, vol. 590, no. 7847, pp. 576–579, Feb. 2021. doi:10.1038/s41586-021-03203-0

[4] A. Kruschinski *et al.*, “Confirming the theoretical foundation of steady-state microbunching”, *Commun. Phys.*, vol. 7, no. 1, May 2024. doi:10.1038/s42005-024-01657-y

[5] M. Sands, “The physics of electron storage rings: an introduction”, Office of Scientific and Technical Information (OSTI), Nov. 1970. doi:10.2172/1453884

[6] A. W. Chao, “Evaluation of beam distribution parameters in an electron storage ring”, *J. Appl. Phys.*, vol. 50, no. 2, pp. 595–598, Feb. 1979. doi:10.1063/1.326070

[7] K. L. F. Bane, Y. Cai, and G. Stupakov, “Threshold studies of the microwave instability in electron storage rings”, *Phys. Rev. Spec. Top. Accel. Beams*, vol. 13, p. 104402, Oct. 2010. doi:10.1103/PhysRevSTAB.13.104402

[8] J. H. Bian *et al.*, “Threshold studies of coherent synchrotron radiation induced microwave instability beyond adiabatic approximation”, *Phys. Rev. Accel. Beams*, vol. 28, p. 074401, Jul. 2025. doi:10.1103/jp4b-1mhy

[9] Y. Cai, “Linear theory of microwave instability in electron storage rings”, *Phys. Rev. Spec. Top. Accel. Beams*, vol. 14, p. 061002, Jun. 2011. doi:10.1103/PhysRevSTAB.14.061002

[10] J. Feikes *et al.*, “Metrology Light Source: The first electron storage ring optimized for generating coherent THz radiation”, *Phys. Rev. Spec. Top. Accel. Beams*, vol. 14, p. 030705, Mar. 2011. doi:10.1103/PhysRevSTAB.14.030705