SYMPLECTIC TRACKING OF DAMPING WIGGLERS USING GENERALIZED GRADIENT REPRESENTATIONS*

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Abstract

Strong wigglers are installed in the storage ring to provide a strong damping effect and thus can make the particle beam reach an equilibrium state quickly. Nevertheless, particle tracking of damping wigglers with strong peak field strengths typically uses the field map analysis approach, which does not account for radiation effects and quantum excitations in ELEGANT. To address this, the Generalized Gradient Expansion (GGExp) method is employed to describe the strong magnetic fields of damping wigglers in storage rings. The GGExp method provides an alternative approach to define the wiggler field, incorporating synchrotron radiation effects with symplectic tracking, and offers benefits for both linear and nonlinear analysis.

INTRODUCTION

Light sources are one of the most powerful tools for scientific research. Although both the synchrotron light source and free electron laser (FEL) have been developed for decades, the poor coherence of the synchrotron radiation and the low repetition rate of the FEL radiation are the main constraints for the high average power. Hence, a Storage-Ring-based Coherent Light Source (SRCLS) [1] is proposed, which combines relatively high coherence with an ultra-high repetition rate and can overcome the disadvantages of these two types of light sources. The SRCLS consists of a photoinjector to produce electron beams with small emittance and a full-energy linac to accelerate them to 1400 MeV. This electron beam then is transported into a storage ring with 12 damping wigglers. The coherent radiation can be generated when the electron beam in the storage ring is extracted to the bypass line, where the electron beam will work in the angular dispersion-induced modulation (ADM) scheme [2]. The degraded electron beam will be reinjected back to the storage ring and damped to the new equilibrium state. To achieve high-repetition-rate FEL radiation, the damping rate should be high. Therefore, a large number of superconducting damping wigglers with high peak field strength are deployed. These wigglers cannot be accurately modeled in the particle tracking code ELEGANT [3] using the CWIGGLER or FTABLE elements. Therefore, the Generalized Gradient Expansion (GGExp) [4, 5] is adopted to reconstruct the real wiggler field to get precise results related to dispersion, emittance, and so on.

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NUMERICAL IMPLEMENTATIONS FOR AN ELLIPTICAL CYLINDRICAL BOUNDARY

Previous simulations show that the dynamic aperture of a light source with damping rings is highly sensitive to the nonlinear characteristics of wiggler transfer maps. In many cases, the computation of single-particle transfer maps for wigglers has relied on idealized wiggler models. However, wiggler transfer maps are generally influenced by fringe-field and high-order multipole effects, which can significantly impact their accuracy. Accounting for these effects requires a detailed and realistic representation of both the interior and fringe magnetic fields, including precise knowledge of higher-order spatial derivatives. In this section, we examine the benefits of calculating these transfer maps using data defined on the surface of a cylinder with an elliptical cross-section.

We have obtained a detailed model of a wiggler field with an elliptical cross-section using data exported from OPERA. Specifically, OPERA provided the three-dimensional magnetic field (**B**) along the entire length of the wiggler, including both the central region and the fringe-field regions. These data were represented on a rectangular mesh, covering the complete magnetic field distribution of the wiggler. By applying polynomial spline interpolation, we extracted the normal magnetic field component (**B**) on the surface of the elliptical cross-section. Subsequently, using these normal components on the surface and employing the scalar potential as an intermediate step, we computed the expansion of the vector potential (**A**) inside the wiggler. This process provides a precise foundation for describing the nonlinear characteristics of the wiggler and its dynamic behavior in damping rings.

In a current-free region, we use a scalar potential ψ that satisfies $\mathbf{B} = \nabla\psi$. To describe the system, we adopt elliptic cylindrical coordinates, which are defined by the intersection of confocal ellipses and confocal hyperbolas. These coordinates are expressed as follows [6]

$$\begin{aligned} x &= f\cosh(u)\cos(v), \\ y &= f\sinh(u)\sin(v), \\ z &= z, \end{aligned} \tag{1}$$

where f is the distance from the origin to the two foci, and u,v are the radial and angular coordinates. Additionally, the scalar potential must satisfy $\nabla^2\psi = 0$. To account for the longitudinal variation, we apply a Fourier transform in z:

$$\tilde{\psi}(r,\phi,k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi(r,\phi,k)e^{-ikz}dz, \tag{2}$$

with $x + iy = re^{i\phi}$. The regular solution (finite on axis) is proportional to the modified Bessel function $I_m(|k|r)$ times angular harmonics. Hence

$$\tilde{\psi}(r,\phi,k) = \sum_{m=0}^{\infty} I_m(|k|r)\left[a_m(k)\cos m\phi + b_m(k)\sin m\phi\right]. \tag{3}$$

For small arguments, the Bessel function can be expanded as

$$I_m(|k|r) = \sum_{n=0}^{\infty} \frac{(|k|r/2)^{2n+m}}{n!\,(n+m)!}. \tag{4}$$

Substituting into the inverse Fourier transform and interchanging summation and integration yields a power series in r. After collecting even and odd powers of k one obtains a representation of the scalar potential as

$$\psi(r,\phi,z) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} \frac{(-1)^n}{4^n n!(n+m)!} r^{2n+m} \times\left[C_{m,s}^{[2n]}(z)\sin m\phi + C_{m,c}^{[2n]}(z)\cos m\phi\right]. \tag{5}$$

The coefficients $C_{m,s}^{[p]}(z)$ and $C_{m,c}^{[p]}(z)$ are related to the on-axis field derivatives via

$$C_{m,s}^{[2n]}(z) = \frac{i^m}{2^{2n}\,n!\,m!}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} k^{2n+m} e^{ikz}\tilde{B}_z(k)\,dk, \tag{6}$$

and analogously for $C_{m,c}^{[2n]}(z)$ using $\tilde{B}_x(k)$. Using $\mathbf{B} = \nabla\psi$ we compute

$$B_r = \frac{\partial\psi}{\partial r}, \qquad B_\phi = \frac{1}{r}\frac{\partial\psi}{\partial\phi}, \qquad B_z = \frac{\partial\psi}{\partial z}. \tag{7}$$

Carrying out the differentiation and re-indexing the sums leads to the following expressions for the magnetic field components in a form suitable for symplectic tracking:

$$\begin{aligned}
B_r &= \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^n m!(2n+m)!}{4^n n!(n+m)!} r^{2n+m-1}\left\{C_{m,s}^{[2n]}(z)\sin m\phi + C_{m,c}^{[2n]}(z)\cos m\phi\right\} + \sum_{n=1}^{\infty}\frac{(-1)^n 2^n}{4^n n! n!} r^{2n-1} C_{0,c}^{[2n]}(z)\\
B_\phi &= \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^n m!(2n+m)!}{4^n n!(n+m)!} r^{2n+m-1}\left\{C_{m,s}^{[2n]}(z)\cos m\phi - C_{m,c}^{[2n]}(z)\sin m\phi\right\}\\
B_z &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(-1)^n m!}{4^n n!(n+m)!} r^{2n+m}\left\{C_{m,s}^{[2n+1]}(z)\sin m\phi + C_{m,c}^{[2n+1]}(z)\cos m\phi\right\}
\end{aligned} \tag{8}$$

GGEXP FIELD REPRESENTATION

Using the theory described above, we demonstrate the field calculation for a superconducting damping wiggler designed for a low-emittance storage ring. Damping wigglers are essential for reducing the transverse emittance and damping time of the beam, but they also introduce strong nonlinear field components that must be accurately characterized. The geometric and magnetic parameters of the wiggler are listed in Table 1. The magnetic field data we used are obtained using the three-dimensional magnetostatic simulation software OPERA, which solves the Maxwell equations via finite-element methods. Because the vacuum chamber inside the wiggler has a relatively large vertical aperture, the simulated field results at positions far from the axis become increasingly unphysical. Consequently, we restrict our analysis to a small but physically relevant region of 40 mm horizontally and 14 mm vertically, which sufficiently covers the transverse beam size including halo.

To obtain a continuous and differentiable representation of the magnetic field on the surface, we compute the field data using cubic spline interpolation. This interpolation method ensures not only continuity of the field values but also smoothness of the first derivatives, which is crucial for accurate particle tracking and for the subsequent extraction of multipole coefficients.

The residuals of the generalized gradient expansion relative to the original field are shown in Fig. 1, and the on-axis

Table 1: Main parameters of the storage ring and the superconducting damping wigglers

Parameter	Value	Unit
Electron beam energy	1400	MeV
Horizontal emittance	800	pm · rad
Period number	14.5	-
Period length	130	mm
Peak field strength	6.31	T
Number of wigglers	12	-
Horizontal chamber size	150	mm
Vertical chamber size	19	mm

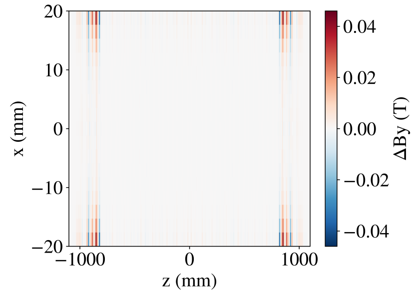

Figure 1: Residual map between the GGExp field and the OPERA data in the $y = 0$ plane.

residuals ΔB_y are shown in Fig. 2. The maximum resid-

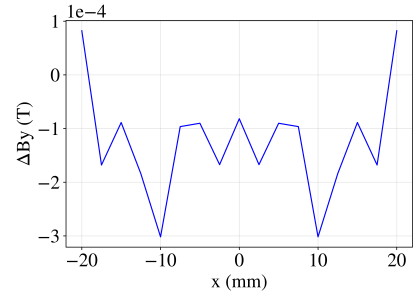

Figure 2: Residual along x at $y = z = 0$.

ual ΔB_y in the plane $y = 0$ is approximately 0.04 T at the two poles, and the residual along x at $y = z = 0$ is about 3×10^{-4} T, which demonstrates the reliability of the surface fitting.

Then, the coefficients for the normal and skew multipole components are written in the SDDS format [7], and symplectic tracking is conducted using the particle tracking code ELEGANT for further simulations.

SYMPLECTIC TRACKING IMPLEMENTATION

To fully account for the nonlinearity of the strong superconducting wigglers, dynamic aperture (DA) analyses are carried out. The lattice and beam parameters follow the descriptions in Ref. [1]. To make a comparison, the damping wigglers are also treated as a series of thin bends to simulate the linear optics and single-particle linear dynamics.

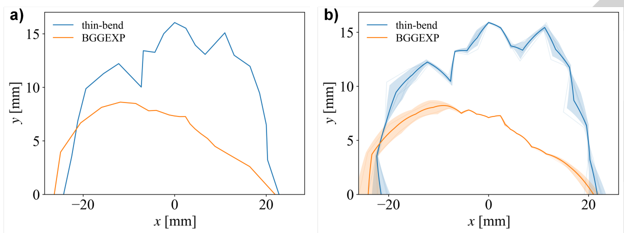

Figure 3: 4D symplectic dynamic aperture tracking in ELEGANT. (a) Lattice without misalignment and (b) with random misalignment. Synchrotron radiation and quantum excitation are not included.

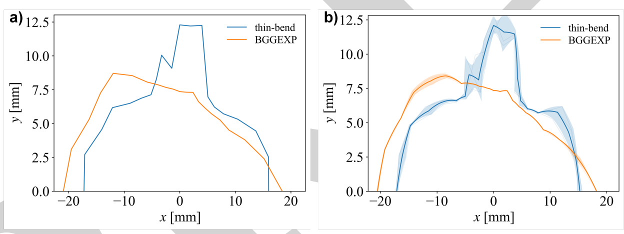

Figure 4: 6D dynamic aperture tracking in ELEGANT. (a) Lattice without misalignment and (b) with random misalignment. Synchrotron radiation and quantum excitation are included.

For the DA tracking with lattice errors, we used an effective residual-error model calibrated using a separate detailed error-and-correction study, which included magnet alignment errors, magnet strength-setting errors, and orbit/optics corrections. After correction, the residual beam-dynamical effect was found to be approximately equivalent to directly applying random transverse magnet offsets with $\sigma_x = \sigma_y = 2\,\mu\text{m}$ without further correction.

Four-dimensional (4D) DA tracking, excluding synchrotron radiation and quantum excitation, is carried out in ELEGANT using both the thin-bend method and the BGGEXP element. As depicted in Fig. 3, the thin-bend method demonstrates a consistently larger DA area compared to BGGEXP, with the latter exhibiting a smaller aperture region under both error-free conditions and when incorporating misalignments. The results indicate that the thin bend approximation is unable to fully account for the nonlinear effects.

When synchrotron radiation and quantum excitation are incorporated—i.e., in six-dimensional (6D) tracking—both methods lead to a reduction in dynamic aperture, as depicted in Fig. 4. Conversely, BGGEXP demonstrates a comparatively modest decrease, maintaining a relatively larger dynamic aperture. Notably, upon the introduction of errors, BGGEXP exhibits a smaller standard deviation, indicating it is less sensitive to errors.

These results indicate that the thin-bend method does not fully account for a more complete field representation, including nonlinearities in the lattice. In contrast, the BGGEXP element effectively incorporates a more complete linear and nonlinear field description, rendering it highly suitable for detailed beam dynamics studies.

CONCLUSION

We applied a generalized gradient expansion (GGExp) representation to model the magnetic field of a superconducting damping wiggler and conducted particle tracking using ELEGANT. Residual analysis confirms that this model provides an efficient field representation for wiggler simulations. Using the BGGEXP element, we performed both 4D and 6D dynamic aperture analyses. In addition, the effect of corrected lattice errors was included through an effective residual-error model calibrated from a separate error-and-correction study. The tracking results obtained with the BGGEXP element exhibit enhanced physics in capturing nonlinear dynamics compared to conventional methods. Future work will focus on a comprehensive high-order chromaticity study.

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