CORRECTION OF THE PATH-LENGTH COORDINATE IN TRACKCPP FOR ACCURATE LONGITUDINAL TRACKING

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Abstract

The Trackcpp application, developed by the Accelerator Physics Group at the Brazilian Synchrotron Light Laboratory, models synchrotron accelerators and beam transport lines to perform six-dimensional particle tracking and optics calculations. The path-length, one of the coordinates, measures the difference between a particle's arrival time, in units of length, and the accelerator cumulative length. This variable is therefore fundamental for describing the interaction of radio-frequency cavities and particles. However, the current implementation leads to limitations when simulating the longitudinal motion in accelerators that operate with non-nominal rf-frequency. This work proposes a modification to the path-length variable that enables accurate longitudinal tracking at arbitrary rf-frequencies.

THEORETICAL BACKGROUND

Computational tools for modeling and simulating particle dynamics are essential for the design, optimization, and operation of particle accelerators. At the Brazilian Synchrotron Light Laboratory (LNLS), the Accelerator Physics Group (FAC) has developed a suite of packages for modeling the SIRIUS light source complex [1]. At the core of these tools is Trackcpp [2], a C++ application that simulates particle tracking in a six-dimensional (6D) coordinate system [3] with symplectic transfer maps based on pass-methods from version 1.3 of the *Accelerator Toolbox* [4]. The 6D state of a particle is

$$\mathbf{x} = (x,\ p_x,\ y,\ p_y,\ \delta,\ z)^T, \tag{1}$$

where x and y are the transverse displacements, p_x and p_y are the normalized transverse momenta, $\delta = (E - E_0)/E_0$ is the relative energy deviation, and z is the path-length deviation.

The equations of motion for a charged particle in an accelerator are derived from a Hamiltonian formulation in which the independent variable is the arc length, s, along a reference orbit rather than time t [3, 5–7]. Within this formalism, the path-length deviation is defined via a canonical transformation that replaces the time t and its canonical momentum by the path-length and the energy deviations [5, 6]:

$$F_2(t,\delta) = \left(\frac{s}{\beta_0} - ct\right)\left(\frac{1}{\beta_0} + \delta\right), \tag{2}$$

which yields

$$z = \frac{\partial F_2}{\partial \delta} = \frac{s}{\beta_0} - ct, \tag{3}$$

where $\beta_0 = v_0/c$ is the relativistic velocity factor and c is the speed of light.

This definition works well when the Hamiltonian is time-independent. When time-dependent components are present, such as the rf-cavities, only keeping track of the z coordinate may lead to loss of information. Considering the model of the cavity gap-voltage is $V_g \sin(2\pi f_{\text{rf}}\, t)$, the phase advance over successive turns in the ring is given by:

$$\phi_n = 2\pi f_{\text{rf}}\,(t_{n+1} - t_n) = 2\pi f_{\text{rf}}\left(\frac{L_0}{\beta_0 c} - \frac{z_{n+1} - z_n}{c}\right) \tag{4}$$

where L_0 is the ring circumference, n is the turn number and we considered $s_{n+1} = s_n + L_0$.

If f_{rf} is equal to the nominal rf-frequency, defined by:

$$f_{\text{rf},0} = \frac{h\,\beta_0\,c}{L_0}, \tag{5}$$

where h is the harmonic number, the first term on the r.h.s. of Eq. (4) reduces to $2\pi h$, which means that the z coordinate can replace t in the argument of the sin function without loss of generality. This is a common approach in modern tracking codes [4, 8]. However, if the accelerator operates at a de-tuned frequency $f_{\text{rf}} = f_{\text{rf},0} + \Delta f$, the phase advance per-turn gains an additional factor, $\Delta\phi$, given by

$$\frac{\Delta\phi}{2\pi} = \Delta f\,\frac{L_0}{\beta_0 c} = -\frac{f_{\text{rf}}}{\beta_0 c}\Lambda, \tag{6}$$

where Λ is the phase slip correction factor in units of traveled distance, given by

$$\Lambda = L_0\left(\frac{f_{\text{rf},0}}{f_{\text{rf}}} - 1\right). \tag{7}$$

This factor has to be taken into account during tracking to correctly reproduce the dynamics.

There are several ways to deal with this issue: one can keep track of the time of flight, as is done in *Elegant* [9], or, as implemented in new versions of the *Accelerator Toolbox* [10], the cavity pass-method can include a turn-dependent phase, $n\Delta\phi_0$. The main drawback of these methods is that the argument of the sin function increases indefinitely with the turn number, which causes loss of precision when tracking for a few thousand turns[1].

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[1] To avoid this problem, in Elegant is possible to change the the time-of-flight of the particles at the cavity with the lowest rf-frequency, so that the integer part of the phase is subtracted from the particle's variable. See Ref. [11] for more details.

This issue motivated us to seek a different choice of coordinate, z', which would incorporate the phase drift $\Delta\phi_0$ accumulated over one turn. Instead of using the canonical transformation of Eq. (2), we can define

$$F_2'(t,\delta) = \left(\frac{f_{\mathrm{rf},0}}{f_{\mathrm{rf}}}\frac{s}{\beta_0} - ct\right)\left(\frac{1}{\beta_0} + \delta\right), \qquad (8)$$

which does not change the definition, hence the physical interpretation, of δ and leads to

$$z' = \frac{f_{\mathrm{rf},0}}{f_{\mathrm{rf}}}\frac{s}{\beta_0} - ct. \qquad (9)$$

In fact, one can verify that with this new definition of the path-length deviation, the phase advance over one turn yields

$$2\pi f_{\mathrm{rf}}\left(t_{n+1} - t_n\right) = 2\pi h + 2\pi f_{\mathrm{rf}}\frac{z'_{n+1} - z'_n}{c}, \qquad (10)$$

for any rf-frequency, which means t can be replaced by z' in the cavity pass-method, without loss of generality. It is also easy to verify that when $f_{\mathrm{rf}} = f_{\mathrm{rf},0}$, the canonical transformation in Eq. (8) reduces to the one in Eq. (2). Finally, one can check that:

$$z'_{n+1} - z'_n = z_{n+1} - z_n + \Lambda. \qquad (11)$$

One advantage of this approach in relation to the others previously mentioned is that here the argument of the sin function does not increase with the number of turns of the simulation, since the particles will tend to oscillate around a small, fixed, value of z'. It is also worth mentioning that this new variable can be used to simulate any cavity or time-dependent component with oscillation frequencies that are integer multiples of the base rf-frequency. The time-dependent elements for which this condition is not met must be dealt differently and are not supported by the current version of Trackcpp.

PATH-LENGTH IN TRACKCPP

Previous Implementation

Before this work, the path-length in Trackcpp was defined according to the canonical transformation in Eq. (2), with $\beta_0 = 1$. As discussed in the previous section, this definition is accurate when operating at the nominal rf-frequency. For non-nominal rf-frequencies, a local workaround existed in the closed orbit (CO) algorithm to deal with the known phase drift: along all iterations to compute the 6D fixed point, the iterative map was augmented with an offset $\theta = (0,0,0,0,0,\Lambda)^T$, to account for the phase drift. This yielded a physically correct synchronous state for the fixed point, $\mathbf{x}^*$, but left the path-length inconsistency unaddressed in all other tracking routines.

Proposed Modification

One possible way to implement the change of coordinate described above would be to modify all pass-methods according to the new Hamiltonian. However, due to historical reasons, the base rf-frequency is a property of the cavity element in Trackcpp, instead of a property of the accelerator being simulated. This technical issue would either increase the run time to retrieve this variable at every pass-method or introduce extra arguments to all pass-methods. For this reason, motivated by the result in Eq. (11), we decided to apply a correction factor, λ, to the path-length deviation only at the time-dependent elements of the ring model, denominated here as *time-aware* elements.

According to Eq. (11), the total correction Λ that makes z equals to z' over one full revolution is:

$$z \leftarrow z' = z + \Lambda. \qquad (12)$$

This way, between two consecutive time-aware elements at positions s_i and s_{i+1}, the correction factor is given by

$$\lambda = \Lambda \cdot \frac{\Delta s}{L_0}, \qquad \Delta s = s_{i+1} - s_i. \qquad (13)$$

With this method, the cumulative correction per turn will always equals Λ, thereby canceling the phase drift for any f_{rf}. With this global correction in place, the workaround term θ in the CO algorithm was no longer necessary and was removed.

RESULTS

To validate the modification, particle tracking simulations were performed in the SIRIUS storage ring model at an rf-frequency shifted by 95 Hz from the nominal value. The longitudinal (z,δ) phase space was recorded at $s_0 = 0$ after each turn and compared between the *Old* (uncorrected) and *New* (corrected) implementations.

Two initial conditions were studied: (i) a particle launched from the CO, $\mathbf{x}_0 = \mathbf{x}^*$; and (ii) a particle with a path-length offset $\kappa = 1$ cm, $\mathbf{x}_0 = \mathbf{x}^* + \kappa\hat{\mathbf{z}}$. Each case was tracked without radiation damping. Since the CO algorithm was already corrected by the θ workaround, $\mathbf{x}^*$ is physically consistent in both versions.

Tracking from the Synchronous State

Figures 1 and 2 show the trajectories for $\mathbf{x}_0 = \mathbf{x}^*$. The particle should remain stationary at $\mathbf{x}^*$. Under the *Old* implementation this does not happens and the trajectory forms a phase-space ellipse centered at $\delta = 0$, the fixed point for the model with nominal rf-frequency. With the *New* implementation, the particle remains stationary to within numerical precision ($|\Delta z|$, $|\Delta\delta| \lesssim 10^{-15}$).

Tracking from a Deviated State

Figures 3 and 4 show the trajectories for $\mathbf{x}_0 = \mathbf{x}^* + \kappa\hat{\mathbf{z}}$. The *Old* implementation again produces an ellipse centered at $\delta = 0$, away from the true fixed-point, while the *New* version correctly produces the elliptic trajectory centered at $\mathbf{x}^*$.

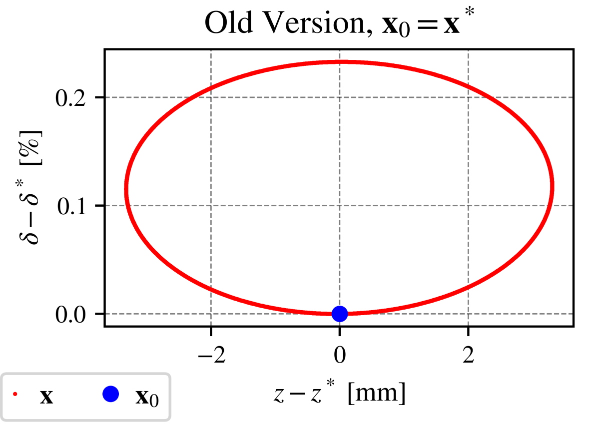

Figure 1: Longitudinal phase-space trajectory for $\mathbf{x}_0 = \mathbf{x}^*$, with the old version of Trackcpp, at $f_{rf} = f_{rf,0} + 95$ Hz.

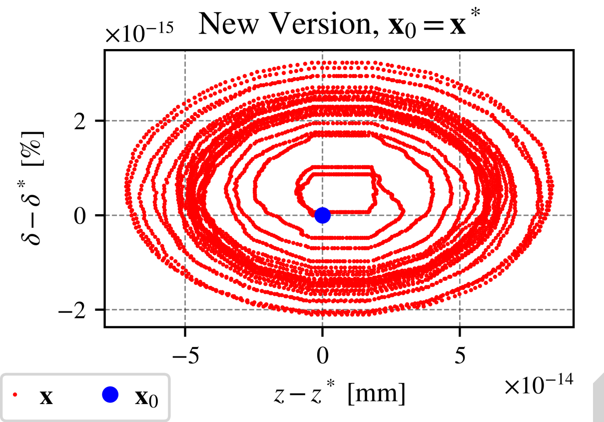

Figure 2: Longitudinal phase-space trajectory for $\mathbf{x}_0 = \mathbf{x}^*$, with the new version of Trackcpp, at $f_{rf} = f_{rf,0} + 95$ Hz.

CONCLUSION

A modification to the path-length variable in Trackcpp was discussed theoretically and implemented to correctly simulate longitudinal dynamics at non-nominal rf-frequencies. Before this work, the definition of z followed a traditional canonical transformation of t, which, allied with an implementation of rf-cavity pass-methods solely based on this new variable, caused an important loss of information whenever the accelerator operated with non-nominal rf-frequency. This lead to wrong time-evolutions of the particle's state, breaking the stationary state of the synchronous particle in all tracking routines.

The proposed change applies an incremental correction term at all time-dependent elements. This globally restores the correct longitudinal dynamics when all time-dependent elements have frequencies multiples of the base rf-frequency and removes the need for a local workaround previously present in the CO algorithm. More complicated time-dependent components, such as long-term wake-functions, will need to be addressed differently when incorporated to Trackcpp.

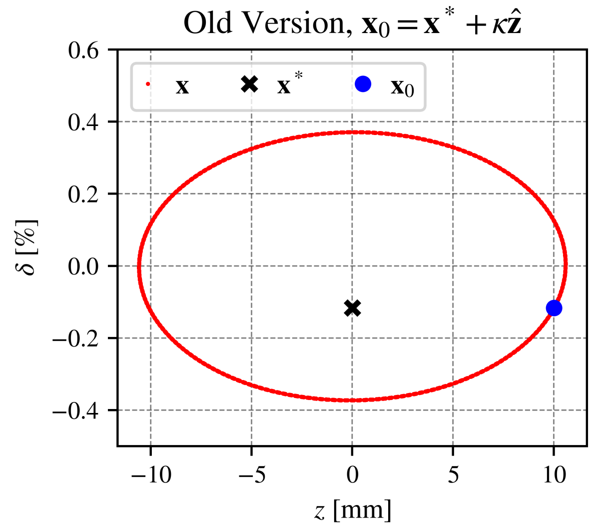

Figure 3: Longitudinal phase-space trajectory for $\mathbf{x}_0 = \mathbf{x}^* + \kappa\hat{\mathbf{z}}$ ($\kappa = 1$ cm), with the old version of Trackcpp, at $f_{rf} = f_{rf,0} + 95$ Hz.

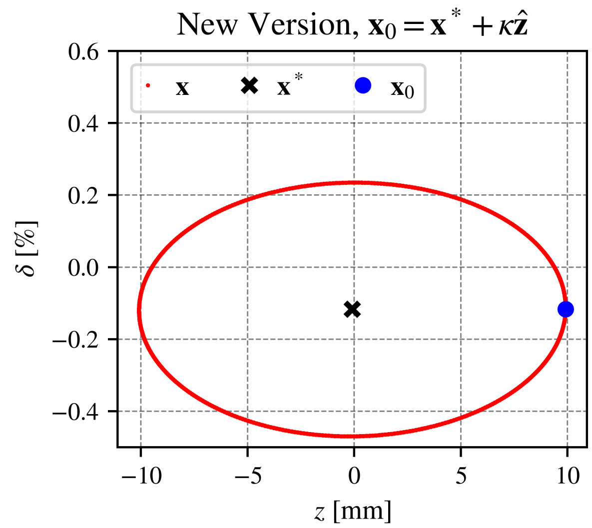

Figure 4: Longitudinal phase-space trajectories for $\mathbf{x}_0 = \mathbf{x}^* + \kappa\hat{\mathbf{z}}$ ($\kappa = 1$ cm), with the new version of Trackcpp, at $f_{rf} = f_{rf,0} + 95$ Hz.

The modification was validated through particle tracking simulations in the SIRIUS storage ring model at non-nominal rf-frequency, demonstrating physically correct synchrotron oscillations centered at the CO for both stationary and displaced initial conditions. The implementation has been merged into Trackcpp main code.

Future work includes implementing the base rf-frequency as a property of the accelerator model, and rewriting all the pass-methods according to the new canonical transformation of the path-length. Also, velocity-dependent effects, which are currently neglected under the ultra-relativistic approximation, will be addressed, enabling more accurate longitudinal simulations of the SIRIUS Booster, where particles are accelerated from 150 MeV to 3 GeV.

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