AN EFFICIENT SYMPLECTIC MODEL FOR RF CAVITIES INCLUDING TRANSVERSE FOCUSING

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Abstract

Tracking through RF cavities is complicated due to the presence of time-varying electric and magnetic fields, which are not amenable to algorithms developed for static fields. Many simulation programs use simplified models that neglect transverse focusing even though it can play an important role at low energies. Other programs track using field maps which can be slow and potentially non-Maxwellian (and therefore not symplectic).

To avoid some of these problems, presented here is an RF cavity model which is symplectic and computationally efficient. Spin tracking is included and transfer maps of arbitrary order can be computed. Additionally, DC solenoid and multipole components can easily be included.

This model is an extension of the work of Rosenzweig and Serafini extended to avoid the ultra-relativistic approximation. The model has been incorporated as part of the Bmad and SciBmad ecosystems of libraries and programs.

INTRODUCTION

Accurate simulation of linac beamlines, requires a tracking model for RF accelerating cavities that captures both the longitudinal energy gain and the transverse focusing effects. At one extreme are models that only have a longitudinal kick and ignore transverse focusing entirely. Another approach is to integrate the particle motion. Integration with the electromagnetic field is accurate but can be computationally expensive and non-symplectic if interpolation is needed to calculate the field. Integration using a split Hamiltonian derived from the field (similar to a "kick-drift-kick" model) can be constructed [1] and while this is symplectic, there is the problem that many time steps may be needed to ensure that the transverse kick is accurate. The root of the problem is that, as shown below, the net first order transverse kick in a cylindrically symmetric RF cavity is zero leaving only the second order "pondermotive" force. However, integration using a field based Hamiltonian will only have a zero first order kick in the limit of an infinite number of steps.

The model described here occupies a practical middle ground. It is based on the work of Rosenzweig and Serafini [2] extended to be symplectic at non ultra-relativistic energies where the normalized particle velocity β is significantly less than one. The model uses a kick-drift-kick Hamiltonian for the longitudinal kick and an integrated Hamiltonian for the transverse kick which ensures that the pondermotive force is properly accounted for. Spin tracking is included in the model and, the model allows for DC solenoid and multipole fields to be superimposed on the cavity.

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This algorithm has been incorporated into the Bmad [3] and SciBmad [4] ecosystems of libraries and programs.

PHASE-SPACE COORDINATES

The phase space canonical coordinates used in this paper are

$$x, P_x, y, P_y, \tau, E \tag{1}$$

where P_x and P_y are the transverse momenta, E is the energy, and τ is

$$\tau = c(t_{\text{ref}} - t) \tag{2}$$

with c being the speed of light, t_{ref} is the reference time, and t is the particle time.

CAVITY MODEL

The cavity structure of length L is modeled as having a number N_c of contiguous cells. Within each cell, the RF is ignored and the particle energy is constant. Kicks are applied at the ends of the cells. The kick points, which are one more than the number of cells, are indexed from zero to N_c. Entrance and exit fringe kicks are applied at the ends of cavity.

EDGE KICKS

At the entrance and exit ends of the cavity, the radial electric fringe field produces a transverse kick. Adapting Rosenzweig and Serafini Eq. (12), the appropriate fringe Hamiltonian H_f is:

$$H_f = \pm \frac{q}{2\,c}\, G\, \cos(\theta_{\text{rf}})\, (x^2 + y^2), \tag{3}$$

where q is the particle charge, G is the voltage gradient, and θ_{rf} is the RF phase (discussed below). The top (minus) sign applies at the entrance end of the cell and the bottom (plus) sign at the exit end.

This Hamilton gives the fringe kicks

$$\begin{aligned}
\Delta P_x &= \mp \frac{q}{c}\, G \cos(\theta_{\text{rf}})\, x, \\
\Delta P_y &= \mp \frac{q}{c}\, G \cos(\theta_{\text{rf}})\, y, \\
\Delta E &= \mp \frac{\pi f q}{c^2}\, G \sin(\theta_{\text{rf}})\, (x^2 + y^2).
\end{aligned} \tag{4}$$

The entrance kick will be defocusing and the exit kick will be focusing for accelerated particles.

The spin precession due to the fringe is calculated using the Thomas-BMT equation with the radial electric field integrated over the fringe being

$$\int_{\text{fringe}} \mathbf{E} = \pm G\, \cos(\theta_{\text{rf}})\, (x, y, 0) \tag{5}$$

LONGITUDINAL ENERGY KICKS

Longitudinal energy kicks are applied at the kick points at the ends of the cells. The Hamiltonian H_E at the n^{th} kick point is

$$H_E(n) = \frac{c\, q\, \kappa(n)\, V_c}{2\,\pi f} \sin(\theta_{\text{rf}}(n)) \tag{6}$$

where $\kappa(n) = 0.5$ at the two end kick points and $\kappa(n) = 1$ at interior kick points. $V_c = V/N_c$ is the voltage across the cell with V being the voltage across the entire cavity. The energy kick at the n^{th} kick point is then

$$\Delta E(n) = q\, \kappa(n)\, V_c \cos(\theta_{\text{rf}}(n)), \tag{7}$$

The formulation of the kick is such that if the phase is the same at all kick points, the total energy gain will be $q\, V \cos(\theta_{\text{rf}})$ as one would expect.

The spin precession is calculated using the Thomas-BMT equation with the integrated longitudinal electric field being

$$\int_{\text{kick}} \mathbf{E}(n) = (0, 0, c \cdot \Delta E(n)/q) \tag{8}$$

RF PHASE

At the n^{th} kick point, the RF phase $\theta_{\text{rf}}(n)$ seen by a particle is

$$\theta_{\text{rf}}(n) = \theta_t + \theta_{\text{ref}}(n) + \theta_0, \tag{9}$$

where $\theta_t = 2\pi f t$ is the phase arising from the particle's arrival time, θ_0 is an overall RF phase set by the user, and $\theta_{\text{ref}}(n)$ is the reference phase.

In theory, this reference phase should be calculated based upon the actual cavity geometry. As calculated in Bmad and SciBmad, since the actual cavity geometry is unknown, the reference phase is set so that a particle entering the cavity on the zero orbit[1], will gain an energy equal to $V\ \cos(\theta_0)$. This construction simplifies for the user the setting of the voltage and phase θ_0. $\theta_{\text{ref}}(n)$ is calculated in Bmad and SciBmad, using the equation

$$\theta_{\text{ref}}(n) = -2\,\pi f \sum_{j=1}^{n} t_c(j) \tag{10}$$

where $t_c(j)$ is the "reference transit time" across the j^{th} cell (cells are indexed from 1 to N_c). $t_c(j)$ is calculated via

$$t_c(j) = \frac{L_c}{c\, \beta_{\text{ref}}(j)} \tag{11}$$

where $L_c = L/N_c$ is the cell length, and $\beta_{\text{ref}}(j)$ is the reference velocity in the j^{th} cell which is computed using a reference energy $E_{\text{ref}}(j)$ in the j^{th} cell of

$$E_{\text{ref}}(j) = E_{\text{ref}}(0) + \sum_{k=0}^{j-1} \frac{\kappa_j}{N_c}\, dE_{\text{ref}} \tag{12}$$

[1] The "zero orbit" is defined by $x = P_x = y = P_y = \tau = 0$ and by the particle having an energy equal to the reference energy. The reference energy is the design-specified energy that particle energies are measured with respect to. In Bmad and SciBmad, reference energies vary from element to element and are typically set to be the desired beam energy.

with $E_{\text{ref}}(0)$ beging the reference energy at the beginning of the cavity, and dE_{ref} being the change in the reference energy across the entire cavity which is set by the user. Typically, dE_{ref} is set to $V\ \cos(\theta_0)$ so that a particle on the zero orbit stays on the zero orbit. However, for studies of cavity voltage and phase errors, this relationship is generally not maintained.

STANDING-WAVE PONDERMOTIVE FOCUSING

The cavity model presented in this paper can be used for both traveling wave and standing-wave cavities. For a traveling-wave cavity, there is a forward-propagating wave and this wave provides both acceleration and the edge kick as discussed above. There is also a transverse force associated with the forward wave but as this force varies as $1/\gamma^2$, where γ is the relativistic gamma factor, this force is ignored in the Rosenzweig and Serafini formulation and will not be considered here.

A standing-wave cavity is modeled as the superposition of a forward-propagating and a backward-propagating wave of equal amplitudes. The backward wave does not contribute to the net acceleration (its longitudinal kick averages to zero for a particle traversing many cells), but the backwards wave produces a second order transverse focusing kick called the pondermotive force [2, 5].

The derivation of the pondermotive force starts with the radial force F_r at radius r in a cylindrically-symmetric cavity written in terms of the longitudinal electric field E_z. Generalizing Rosenzweig and Serafini Eq. (1) to sub ultra-relativistic energies (See Hartman & Rosenzweig [5] Eq. (17)) gives:

$$F_r = -\frac{q\, r}{4}\, (1 + \beta^2)\, \frac{dE_z}{dz} \tag{13}$$

where d/dz is the total derivative with respect to z. The transverse force is assumed to be small so that a perturbation expansion can be used. Since the field drops to zero at the ends of the cavity, the first order integrated kick, which is the kick integrated at constant r, will be zero.

$$\begin{aligned}\int dz\, F_r &= -\frac{q\, r}{4}\, (1 + \beta^2)\, (E_z(\text{exit}) - E_z(\text{entrance})) \\ &= 0\end{aligned} \tag{14}$$

Going to second order, a particle's transverse motion oscillates around some average r and this oscillation, in phase with the backwards wave, leads to an average focusing force – the pondermotive force – which is given by Rosenzweig and Serafini Eq. (5) in the ultra-relativistic limit.

To maintain symplecticity at lower energies, the appropriate Hamiltonian averaged over a cell corresponding to Rosenzweig and Serafini Eq. (5) with $\eta(\theta) = 1$ (no RF harmonics) is:

$$H_p = \frac{L_c\, q^2 G^2}{16\, c^2 E}\, (x^2 + y^2), \tag{15}$$

With the cavity model of this paper, the pondermotive kick is applied at each kick point. The resulting phase-space kicks are

$$\Delta P_x = -\frac{\kappa(n)\, L_c\, q^2 G^2}{8\, c^2 E}\, x, \tag{16}$$

$$\Delta P_y = -\frac{\kappa(n)\, L_c\, q^2 G^2}{8\, c^2 E}\, y, \tag{17}$$

$$\Delta \tau = -\frac{\kappa(n)\, L_c\, q^2 G^2}{16\, c^2 E^2}\,(x^2 + y^2). \tag{18}$$

Several features are noteworthy. The kick is always focusing and independent of RF phase (since the phase of the backward-wave that the particle sees is being averaged over). The kick is quadratic in G (that is, a second order effect), analogous to the net focusing produced by alternating-gradient structures. The longitudinal time shift $\Delta\tau$ arises from the transverse oscillation that the backward wave induces: the sinusoidal transverse motion slightly delays the particle, reducing τ.

Since the pondermotive force is relatively weak, the effect on spin is ignored.

SOLENOID AND MULTIPOLE COMPONENTS

A DC solenoid field component can be superimposed on the cavity. In this case, the free drifts within each cell are replaced by tracking through a uniform solenoid field. DC multipoles can also be added to the model.

In the model, as implemented by Bmad and SciBmad, a cavity lattice element has an "active" length over which there is an RF field. The element length can be greater than this and this allows for solenoid and multipole fields to extend past the RF cavity fields.

CONCLUSION

A model for RF cavities which is an extension of the work of Rosenzweig and Serafini has been described that is applicable for sub ultra-relativistic beams.

The model is suitable for use with both rings and linacs. The model is computationally efficient: for high energy beams, a cavity with a single-cell ($N_c = 1$) is often sufficient. Additionally, transfer maps to arbitrary order can easily be computed.

Spin tracking is included in the model and DC solenoid and multipole components can be added. This model has been implemented in both the Bmad and SciBmad ecosystems of simulation programs and libraries.

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