TRACKING-BASED LINEAR SYMPLECTIC PARAMETERIZATION OF INSERTION DEVICES[*]

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Abstract

Accurate linear models of insertion devices (IDs) are essential for optics correction, coupling control, and feed-forward strategies in modern light sources. We present a systematic framework that builds a linear transverse symplectic parameterization, including leading-order chromatic effects, directly from Radia's Runge–Kutta (RK) particle tracking. We compare the resulting matrices with those obtained using the standard kick-maps approach for planar and elliptic IDs. The resulting parameterized symplectic models are well suited for constructing feed-forward tables across ID operating ranges.

INTRODUCTION

IDs can have a significant impact on both linear and nonlinear beam dynamics in modern light sources, depending on the accelerator lattice and operating energy. In particular, correction of linear ID effects during changes of ID parameters is a routine operation performed through feed-forward tables. This ensures stable tunes and preserves beam sizes at other beamlines.

The magnetic fields in IDs can be accurately modeled, including realistic geometries and materials, using specialized software, e.g. Radia [1, 2]. Linear effects can be modeled based on a special potential constructed from magnetic field over one ID period [3]. However, this approach relies on several assumptions and does not account for matching sections. To assess the importance of realistic geometries and magnetic fields, a linear-model construction method based on Radia's RK integrator was developed [4]. Different linear models for Elettra 2.0 [5] are compared with respect to their impact on beam optics for both linearly polarized undulators (LPUs) and elliptically polarized ones (EPUs). The EU100 EPU (APPLE-II), with period length $\ell_p = 100\,\text{mm}$, total number of periods $n_p = 40$, and $(k_x, k_y) = (3.9, 6.6)$ with 31 mm gap (see Ref. [6] for details), is used to compare the models and to illustrate the effects on beam optics.

ID TRANSPORT MODELING

Most of the Elettra 2.0 IDs have detailed Radia models that allow accurate magnetic field computation and have been validated previously [7, 8]. Given the magnetic field, the standard procedure is to construct a special potential [3] used to express the transverse angle kicks over one ID period, describing both linear and nonlinear motion. An ID can be modeled by a single thin kick, or a more accurate one-kick-per-period computation can be performed.

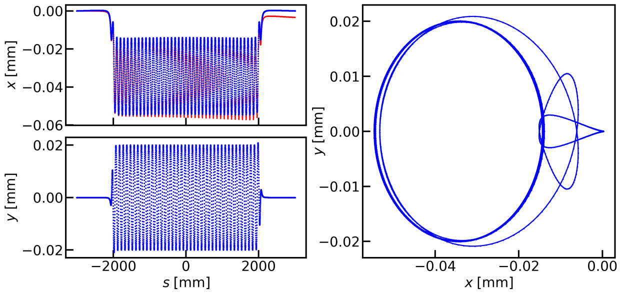

Figure 1: Particle trajectory in the EU100 (APPLE-II) insertion device for unmatched (red) and matched (blue) orbits.

Direct trajectory integration using initial conditions close to the reference trajectory (see Figure 1) can also be used to construct a linear transport matrix in the form of a thin-lens insertion. Other approaches, primarily aimed at nonlinear dynamics, can be constructed by employing parametric field models or by fitting basis functions to field data [9–11].

Elleaume's Potential and Transverse Kicks

Under the assumption that the first-order field integrals vanish, the leading angular kick appears at second order and can be derived from the following potential function:

$$\Phi(x, y, s) = \left(\int_{-\infty}^{s} B_x(x, y, s')\, ds'\right)^2 + \left(\int_{-\infty}^{s} B_y(x, y, s')\, ds'\right)^2,$$

where (B_x, B_y, B_z) are the magnetic field components generated by the ID and available for evaluation at any point in space.

For a periodic ID, the one-period potential is

$$U(x, y) = \oint \Phi(x, y, s)\, ds,$$

and the changes in the transverse slopes $x' = dx/ds$ and $y' = dy/ds$ over one period are

$$\Delta x' = -\frac{\alpha^2}{2}\frac{\partial U(x, y)}{\partial x}, \qquad \Delta y' = -\frac{\alpha^2}{2}\frac{\partial U(x, y)}{\partial y},$$

where $\alpha = e/\gamma mc$.

In practice, instead of evaluating the integral for $U(x, y)$ directly, a truncated Fourier representation of the periodic magnetic field is used. For a fixed transverse position, one or more harmonics of the field components are computed over one period from numerical field samples. The one-period potential is then approximated from the amplitudes of these harmonics.

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Radia's RK Integrator

Radia computes magnetic fields and potentials at arbitrary points in space. These can be used to numerically solve the equations of motion for arbitrary initial conditions. Radia also provides a built-in Runge–Kutta integrator to compute full trajectories. In general, a zero initial condition at the device entrance is not mapped to zero at the exit, mainly because the mean orbit through the device is not zero. A correction enforcing a zero-to-zero mapping can be obtained by solving a fixed-point problem, but this effect is small and its impact on the transport matrix can be neglected.

To be used in tracking codes, the thin ID mapping should be expressed in canonical coordinates. Since Radia integrates the equations using slopes, the initial canonical momenta (p_x, p_y) are first transformed to slopes (x', y'):

$$x' = \frac{p_x}{\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}}, \quad y' = \frac{p_y}{\sqrt{(1+\delta)^2 - p_x^2 - p_y^2}},$$

where δ is the longitudinal momentum deviation, assuming zero field at the ID entrance and exit. The inverse transformation is applied at the device exit. The thin-insertion mapping is then obtained by applying negative drift transformations. For the potential model, drift–kick–drift steps are performed in slope coordinates for each period, with the above transformations applied at the ID entrance and exit. The mapping is already in thin-insertion form if only one kick is used for the full device.

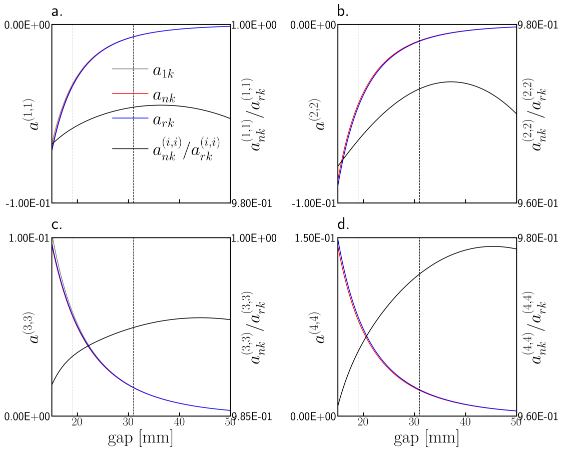

Figure 2: Main diagonal elements vs ID gap size comparison between different linear models.

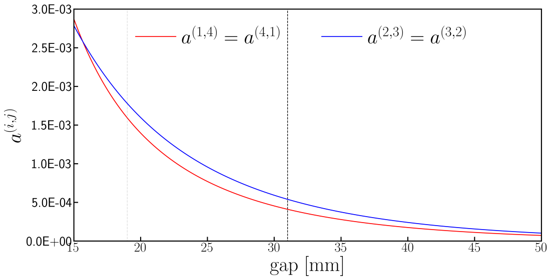

Figure 3: Skew diagonal elements vs ID gap size.

LINEAR ID MODEL

Given a nonlinear thin-insertion mapping, the corresponding transport matrix can be extracted by tracking several initial conditions close to the origin. The resulting matrix is generally not exactly symplectic, but it is expected to be close to a symplectic one. A standard symplectification procedure can then be applied to obtain a symplectic matrix M that can be used to model the ID effect on the linear optics. The exponential representation $M = \exp(SA)$, where S is the symplectic form and A is a symmetric matrix, can then be computed, yielding a redundancy-free representation and providing a direct link to the corresponding quadratic Hamiltonian.

Using a diagonal matrix of small initial deviations $\Lambda(\varepsilon)$, define $M^+ = \varphi(\Lambda(+\varepsilon))$ and $M^- = \varphi(\Lambda(-\varepsilon))$. The transport matrix is then constructed

$$M = \frac{1}{2\varepsilon}\left(M^+ - M^-\right)^T,$$

and the corresponding exponent is $A = -S\log(M)$.

To include the leading order chromatic effects, the transport matrix is factorized as

$$M(\delta) = \exp(SA)\exp(\delta SB),$$

where the symmetric matrix B describes the leading-order chromatic contribution. A leading-order expansion gives

$$M(\delta) = M(0) + (dM/d\delta)_{\delta=0}\,\delta + \cdots = M(0)\left(I + \delta SB + \cdots\right),$$

which yields $B = -S\exp(-SA)\,(dM/d\delta)_{\delta=0}$. The derivative can be evaluated by central finite differences,

$$(dM/d\delta)_{\delta=0} = \frac{M(+\Delta) - M(-\Delta)}{2\Delta},$$

for a small momentum deviation Δ. The resulting matrix B is expected to be close to symmetric, and $\hat{B} = 1/2(B + B^T)$ can be used to enforce symmetry.

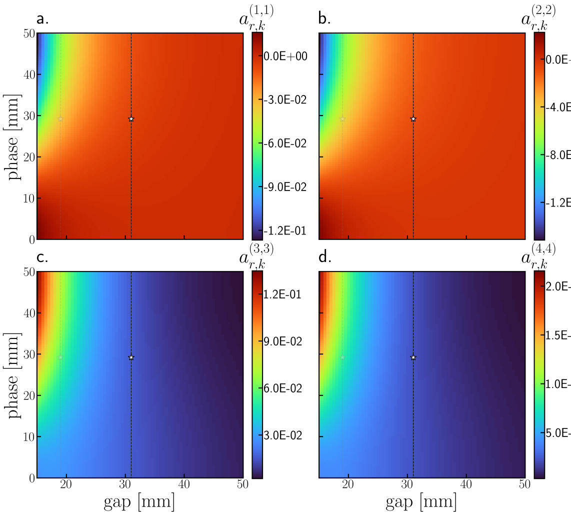

Figure 4: Main diagonal elements vs ID gap and phase scan.

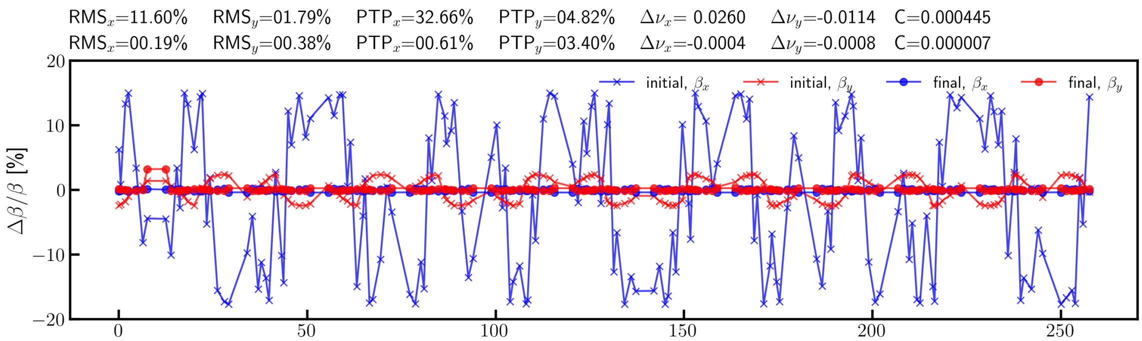

Figure 5: Beta-beating generated by EU100 before and after correction.

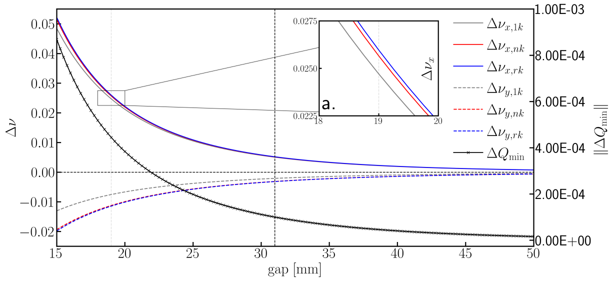

Figure 6: Tune shift vs gap size and minimal tune distance.

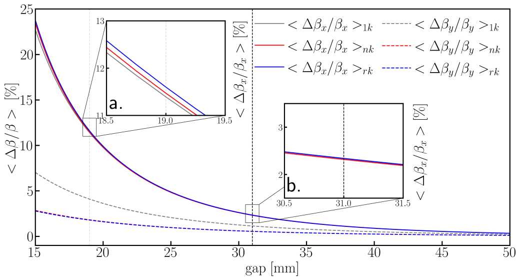

Figure 7: Beta-beating vs gap size.

COMPARISON OF LINEAR MODELS

Linear models constructed from tracking based on the potential (a single kick and one kick per period, denoted 1K and NK, respectively) or on full RK tracking were compared for several Elettra 2.0 IDs, including LPU, EPU, and Figure-8 devices. Overall, the observed difference in the main diagonal elements of the A matrix varies from device to device, ranging from a few percent to several tens of percent for short IDs. This results in tune-shift differences of 10^{-4} and an RMS beta-beating difference of about 0.1 %. Both effects are negligible for Elettra 2.0.

Examples of the diagonal elements of the A matrix for the EU100 are shown in Figure 2 as a function of ID gap. The discrepancy between the NK and RK models is within a few percent. Only the RK model contains linear coupling, represented by the skew diagonal elements, but their absolute values are several orders of magnitude smaller than those of the main diagonal elements (see Figure 3). For the RK model, the diagonal elements as functions of ID gap and phase are shown in Figure 4, demonstrating a smooth dependence on these parameters.

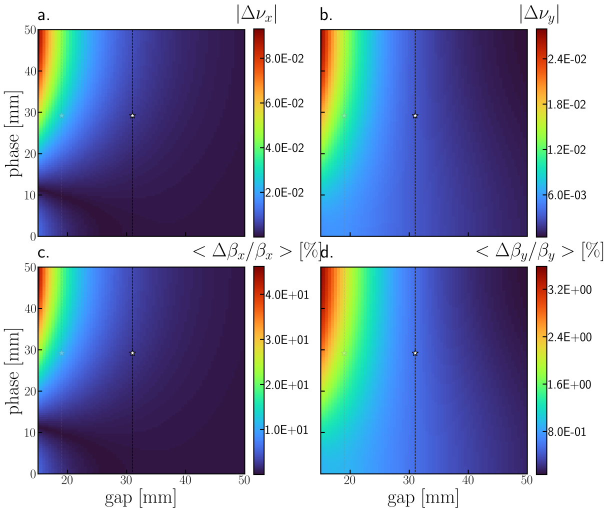

Figure 8: Tune shift and RMS beta-beating vs ID parameters.

Effects on the main linear parameters, including tune shifts, beta-beating, betatron coupling, and dispersion, were studied for the Elettra 2.0 lattice. An example of the optics distortion induced by the EU100 is shown in Figure 5. The most notable effects are tune shifts and beta-beating (see Figures 6 and 7), while coupling, predicted only by the RK model, is small and can be neglected. The effects on dispersion and chromaticity after correction are also negligible. The 1K model was found to produce inaccurate results, with up to 30 % tune-shift mismatch and a factor-of-two mismatch in RMS beta-beating in the vertical plane, while the NK model remains within the required Elettra 2.0 target tolerances. Figure 8 shows the relevant optics parameters for the RK model as functions of ID gap and phase.

CONCLUSION

Linear ID models based on Elleaume's potential (a single kick and one kick per period) and on the RK integrator were investigated for different Elettra 2.0 IDs. Differences in the resulting optics distortion between the NK and RK models were found to be small and within Elettra 2.0 tolerances (RMS beta-beating < 5 %). The RK model also includes linear coupling, which is small but may become relevant for future light sources.

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