BEAM TRAPPING IN STABLE RESONANT ISLANDS OF THE LONGITUDINAL PHASE SPACE BY RF PHASE MODULATION IN ELECTRON RINGS

A. Bazzani[1,2], F. Capoani[3], M. Giovannozzi[3*], E. Nori[1]
[1]Dipartimento di Fisica e Astronomia, Università di Bologna, via Irnerio 46, Bologna, Italy
[2]INFN Sezione di Bologna, Viale Berti Pichat 6/2, Bologna, Italy
[3]Beams Department, CERN, Geneva, Switzerland

Abstract

An RF phase modulation whose frequency is set close to a resonant condition with the synchrotron tune can be used to form resonance islands in the longitudinal phase space. A slow variation of the modulation amplitude and frequency can be performed to trap particles inside the island structure and to realise adiabatic transport, according to adiabatic theory for Hamiltonian systems. However, when dealing with electron beams, energy damping and quantum excitation effects cannot be neglected, and the adiabatic trapping into resonance can be described as an isothermal transformation of an ensemble of particles that evolves according to an adiabatically modulated Hamiltonian. In this study, we establish the theoretical framework for describing longitudinal motion under slow modulation of the RF phase using stochastic dynamical system theory. We also present numerical simulations with parameters of realistic lepton rings to characterise beam splitting between the core region and a 1 : 1 resonant island. Special attention has been given to the thermal properties of the two beamlets, which define their equilibrium emittances and their lifetimes.

INTRODUCTION

Several non-linear beam manipulation schemes have been proposed in recent years, the most notable example being Multi-Turn Extraction (MTE) at the CERN Proton Synchrotron (PS) [1–3]. These techniques aim to control the characteristics of the transverse beam distribution through adiabatic changes in the parameters of the ring or specific devices of the ring [4–7]. Their theoretical description is traditionally based on Hamiltonian theory, which is well suited for hadron beams. More recently, non-linear beam dynamics has also become a topic of experimental investigation at synchrotron light sources [8–22], complemented by numerical studies [23, 24]. The Hamiltonian formalism is no longer adequate and an alternative framework must be adopted. In this work, we consider a few elementary manipulations that involve non-linear beam dynamics and employ the appropriate approach to interpret the corresponding simulation results. The model parameters are taken from the FCC-ee configurations [25] in view of possible future applications. We note that an experimental study of a model similar to that considered here has been presented in Ref. [26].

* massimo.giovannozzi@cern.ch

THE MODEL

Closely following the treatment used in Ref. [27] one starts by considering an electric field of the form $\mathcal{E} = \mathcal{E}_0 \sin(h\omega_{\rm rev} t + \phi_{\rm s})$, where h is the harmonic number, $\omega_{\rm rev}$ the angular revolution frequency and $\phi_{\rm s}$ is the phase of the synchronous particle. The energy gain is given by

$$\dot{E}_{\rm s} = \frac{\omega_{\rm rev}}{2\pi} eV \sin\phi_{\rm s} \qquad \dot{E} = \frac{\omega}{2\pi} eV \sin\phi \tag{1}$$

for the synchronous particle and the generic particle, respectively, and the dot indicates the time derivative. Introducing an alternative variable δ, one obtains ($\beta_{\rm s} = v_{\rm s}/c$)

$$\delta = \frac{\Delta p}{p_{\rm s}} \qquad \dot{\delta} = \frac{\omega_{\rm rev} eV}{2\pi E_{\rm s} \beta_{\rm s}^2} \left(\sin\phi - \sin\phi_{\rm s}\right) , \tag{2}$$

and the dynamics is described by the Hamiltonian

$$\begin{aligned} \mathcal{H}_0\left(\phi, \delta\right) = \frac{1}{2} h \eta_{\rm s} \omega_{\rm rev} \delta^2 + \frac{\omega_{\rm rev} eV}{2\pi E_{\rm s} \beta_{\rm s}^2} [\cos\phi - \cos\phi_{\rm s} + \\ + (\phi - \phi_{\rm s}) \sin\phi_{\rm s}] \qquad \eta_{\rm s} = \gamma_{\rm s}^{-2} - \gamma_{\rm t}^{-2} . \end{aligned} \tag{3}$$

If a modulation of the RF phase, of the form $a \sin(\omega_{\rm m} t + \psi_0)$, where $\omega_{\rm m} = \nu_{\rm m} \omega_{\rm rev}$ is applied, the equation of motion for ϕ is modified and can be derived from

$$\mathcal{H}\left(\phi, \delta, t\right) = \mathcal{H}_0\left(\phi, \delta\right) + a\omega_{\rm m} \cos(\omega_{\rm m} t + \psi_0) \delta . \tag{4}$$

Whenever energy damping is included, if $P_\gamma = \frac{cC_\gamma}{2\pi} \frac{E^4}{\rho^2}$ ($C_\gamma = 8.85 \times 10^{-5}\,{\rm m\,GeV^{-3}}$ for electrons) is the instantaneous power radiated, the energy loss per turn is

$$U_0 = \int_0^{T_{\rm rev}} {\rm d}t\, P_\gamma = \frac{1}{c} \oint {\rm d}s\, P_\gamma = C_\gamma \frac{E^4}{\rho} , \tag{5}$$

where $T_{\rm rev}$ is the revolution period, ρ the bending radius, and we assume the case of an iso-magnetic ring.

If the RF cavity simply restores the energy lost due to damping, the bucket is stationary, i.e. $\phi_{\rm s} = \pi$, and the damping contribution to the variation of δ is

$$\dot{\delta} = -2 \frac{\alpha_{\rm E}}{\beta_{\rm s}^2} \delta \qquad \alpha_{\rm E} = \frac{U_0}{2 T_{\rm rev} E_{\rm s}} \left(2 + \mathcal{D}\right) , \tag{6}$$

with $\mathcal{D}$ the damping partition number defined as

$$\mathcal{D} = \frac{1}{2\pi} \oint {\rm d}s\, D_x(s) \left[\frac{1}{\rho^2} + 2K(s)\right]_{\rm dipoles} , \tag{7}$$

$D_x(s)$ being the dispersion and K the quadrupole gradient.

Under the assumption of the existence of a maximum entropy stationary equilibrium, the addition of quantum excitation to the dynamics with damping can be performed using a stochastic fluctuating term that satisfies the fluctuation-dissipation Einstein relation to simulate the effect of a thermal bath, namely

$$D = \sqrt{2T\gamma m}\,, \tag{8}$$

where γ stands for the damping coefficient, m is the mass, and T the system temperature. In our case, we have

$$m = \frac{1}{h|\eta_\mathrm{s}|\omega_\mathrm{rev}} \qquad \gamma = 2\frac{\alpha_\mathrm{E}}{\beta_\mathrm{s}^2} \qquad T = \frac{\langle\delta^2\rangle}{m} \tag{9}$$

and we obtain

$$T = \frac{1}{m}\left(\frac{\sigma_\mathrm{E}}{E_\mathrm{s}\beta_\mathrm{s}^2}\right)^2 = \frac{1}{m}\frac{C_\mathrm{q}\gamma_\mathrm{s}^2\langle 1/\rho^3\rangle}{(2+\mathcal{D})\,\beta_\mathrm{s}^4\langle 1/\rho^2\rangle}\,, \tag{10}$$

where $C_\mathrm{q} = 3.83\times 10^{-13}$ m, and for iso-magnetic rings

$$D = \frac{\gamma_\mathrm{s}}{\beta_\mathrm{s}^3}\sqrt{\frac{\alpha_\mathrm{E}C_\mathrm{q}}{\rho}} \qquad\qquad T = \frac{\gamma_\mathrm{s}^2 h|\eta_\mathrm{s}|\omega_\mathrm{rev}}{2\beta_\mathrm{s}^4}\frac{C_\mathrm{q}}{\rho}\,. \tag{11}$$

All effects can be gathered in the following set of equations

$$\begin{aligned}\dot\phi &= h\eta_\mathrm{s}\omega_\mathrm{rev}\delta + a\omega_\mathrm{m}\cos(\omega_\mathrm{m}t+\psi_0)\\ \dot\delta &= \omega_\mathrm{rev}\frac{eV}{2\pi E_\mathrm{s}\beta_\mathrm{s}^2}\sin\phi - 2\frac{\alpha_\mathrm{E}}{\beta_\mathrm{s}^2}\delta + \frac{\gamma_\mathrm{s}}{\beta_\mathrm{s}^3}\sqrt{\frac{\alpha_\mathrm{E}C_\mathrm{q}}{\rho}}\xi(t)\,.\end{aligned} \tag{12}$$

where $\xi(t)$ is a stationary random process with zero mean and correlation $\langle\xi(t+s)\xi(t)\rangle = \frac{1}{2\tau}\exp\left(-\frac{s}{\tau}\right)$ so that we recover the white noise limit for $\tau\to 0$.

THEORETICAL ASPECTS

The origin is an elliptic fixed point of the Hamiltonian (4) when $a = 0$ and, in the case of the $1:1$ resonance, $\omega_\mathrm{s} = \omega_\mathrm{rev}\sqrt{\frac{heV|\eta_\mathrm{s}|}{2\pi E_\mathrm{s}\beta_\mathrm{s}^2}} \simeq \omega_\mathrm{m}$. Using a rotating frame and averaging theorems, the phase space of the Hamiltonian (4) is locally described by the normal form Hamiltonian ($2\mathcal{J} = P^2 + Q^2$, (Q,P) being the normal form variables)

$$\begin{aligned}\bar{\mathcal{H}}(Q,P,\lambda) &= \hat{\mathcal{H}}\left(\mathcal{J}(Q,P),\lambda\right) + \hat a(\lambda)P\\ \hat{\mathcal{H}}\left(\mathcal{J}(Q,P),\lambda\right) &= \Delta\omega(\lambda)\mathcal{J} + \sum_{k\ge 2}\omega_k\mathcal{J}(Q,P)^k\,,\end{aligned} \tag{13}$$

where $\Delta\omega = \omega_\mathrm{s} - \omega_\mathrm{m}(\lambda)$, ω_k are the non-linear detuning coefficients, and $\lambda = \epsilon t$ is the slow time that introduces a modulation of the amplitude $\hat a(\lambda) \simeq a(\lambda)\omega_\mathrm{s}\omega_\mathrm{m}(\lambda)$. The phase space of Eq. (13) initially has a stable fixed point whose position depends on $\hat a$. A bifurcation phenomenon appears at a critical value of $\Delta\omega$, when a new pair of stable and unstable fixed points is generated (see Fig. 1).

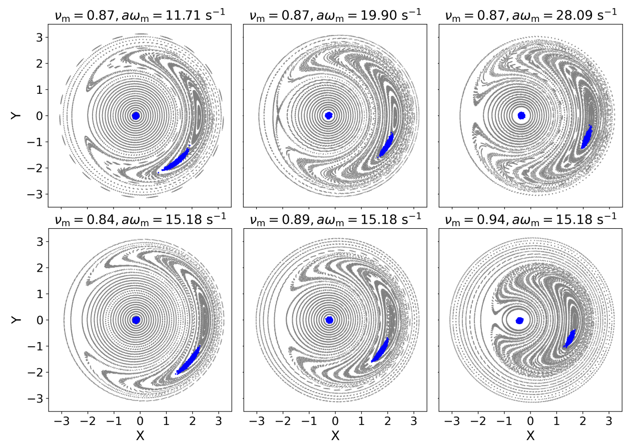

Figure 1: Phase-space portraits of the Hamiltonian (4) using $X = \sqrt{2I}\cos\theta, Y = \sqrt{2I}\sin\theta$ vs. ν_m and $a\omega_\mathrm{m}$. Equilibrium distributions (blue) from Eqs. (12) are also shown. The fixed point displacement due to damping is clearly visible.

Assuming Stratonovich's interpretation of stochastic differential equations and retaining the leading terms of the thermal bath, we get the equations of motion

$$\begin{aligned}\dot Q &= \hat{\mathcal{H}}'\left(\mathcal{J}(Q,P),\lambda\right)P + \hat a(\lambda)\\ \dot P &= -\hat{\mathcal{H}}'\left(\mathcal{J}(Q,P),\lambda\right)Q - \gamma P + \sqrt{\frac{2T\gamma}{\omega_\mathrm{s}}}\xi(t)\,.\end{aligned} \tag{14}$$

In each phase space-region containing an elliptic fixed point, it is possible to introduce action-angle variables $(\theta(Q,P,\lambda), I(Q,P,\lambda))$ and to apply an averaging theorem on the angle variable so that the action dynamics reads

$$\dot I \approx -\frac{\gamma}{2\omega_\mathrm{s}}\left\langle\left(\frac{\partial Q}{\partial\theta}\right)^2\right\rangle\hat{\mathcal{H}}'(I,\lambda) + \sqrt{\frac{T\gamma}{\omega_\mathrm{s}}\left\langle\left(\frac{\partial Q}{\partial\theta}\right)^2\right\rangle}\xi(t) \tag{15}$$

where $\langle\ \rangle$ denotes the angle average, and the perturbation term is estimated using perturbation theory, the estimate being correct far from a separatrix. According to Eq. (15), the Fokker-Planck (FP) equation in the action variable reads

$$\frac{\partial\rho}{\partial t} = \frac{\gamma}{2\omega_\mathrm{s}}\frac{\partial}{\partial I}\left\langle\left(\frac{\partial Q}{\partial\theta}\right)^2\right\rangle\left[\hat{\mathcal{H}}'(I,\lambda)\rho + T\frac{\partial\rho}{\partial I}\right]\,, \tag{16}$$

and ρ tends to the Maxwell-Boltzmann (MB) distribution

$$\rho_\mathrm{eq}(I,\lambda) = \rho_\mathrm{MB}(I,\lambda) \propto \exp\left(-\frac{\hat{\mathcal{H}}(I,\lambda)}{T}\right)\,. \tag{17}$$

RESULTS OF NUMERICAL SIMULATIONS

Equations (12) have been solved using a symplectic integrator for the Hamiltonian part, modified to include damping and stochastic components. The longitudinal beam dynamics of the Z configuration of the FCC-ee [25], which features the lowest damping rate ($\gamma = 5.65\ \mathrm{s}^{-1}$) and the lowest value of $T = 2.83\times 10^{-3}\ \mathrm{s}^{-1}$ among the four FCC-ee configurations, has been studied: in the absence of RF phase modulation, the dynamics around the unique fixed point at the origin is well known: an initial distribution evolves towards ρ_{MB} and an equilibrium emittance exists.

When the modulation of the RF phase is added, the topology of the phase space is deeply affected as shown in Fig. 1 where the phase-space portraits of the Hamiltonian (4) are shown (grey lines) for some values of the modulation tune and amplitude. Three fixed points are present, two elliptical and one hyperbolic, and a stable island corresponding to the $1:1$ resonance is clearly visible. When damping and noise are added, the situation changes again as shown by the distribution of some initial conditions at equilibrium (blue points). One set of points remains close to the origin, indicating that the fixed point in the presence of damping and noise is barely affected. However, those located at the amplitude of the stable fixed point in the island clearly show a phase shift with respect to the situation for the Hamiltonian system, whereas the amplitude is approximately unchanged. This shows that the stable fixed point is affected by damping and that the simple Hamiltonian system is not sufficient to describe the observed dynamics with damping and noise.

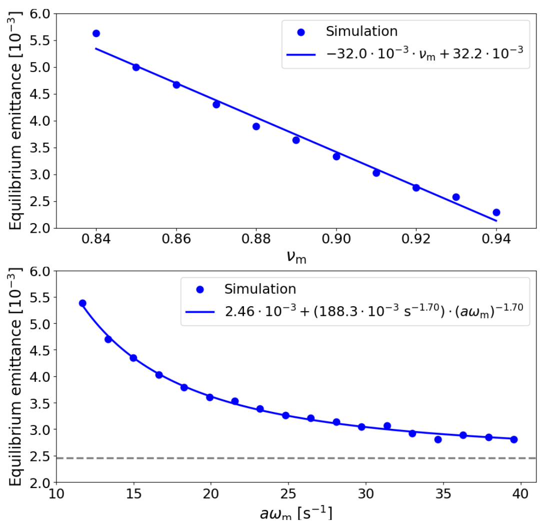

Figure 2: Equilibrium emittance in island vs. ν_{m} (at constant amplitude) and amplitude ($a\omega_{\mathrm{m}}$) together with fit functions.

Detailed investigations of the properties of the equilibrium distribution for the dynamics close to the centre or to the stable island have been carried out and revealed a number of interesting features. Close to the centre, the equilibrium emittance is barely affected by the properties of modulation of the RF phase, whereas the equilibrium emittance strongly depends on ν_{m} and $a\omega_{\mathrm{m}}$, as shown in Fig. 2. A linear dependence on ν_{m} is found, whereas a power-law is visible for the scaling with $a\omega_{\mathrm{m}}$ of the equilibrium emittance.

The third and fourth moments have been studied and revealed that for the case of the centre, they follow the behaviour of a Gaussian distribution, except when the island is moved to low amplitude and in that case a departure from the pure Gaussian case is observed. In contrast, the distribution in the stable island always features non-Gaussian third and fourth moments (see Fig. 3), which are a consequence of the non-linear dynamics around the fixed point.

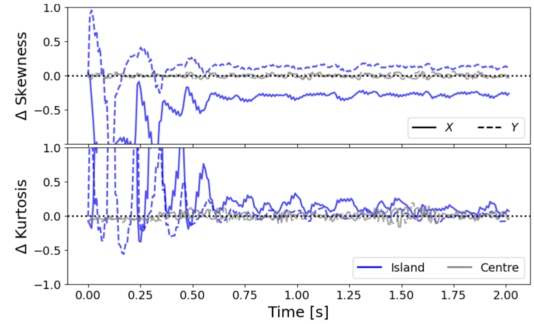

Figure 3: Evolution of selected moments of the simulated distributions, expressed as differences to the corresponding moments of a Gaussian with the same covariance matrix.

As a last aspect, the time to reach the equilibrium emittance in the centre and islands has been studied as a function of the parameters of the RF phase modulation and the results are shown in Fig. 4 as a function of ν_{m} and ω_{m}, and in all cases the functional dependence is well represented by a straight line. For the centre, the values remain close to $1/\gamma$, with a very weak dependence on the parameters, and the relaxation time is shorter than for the case of the island).

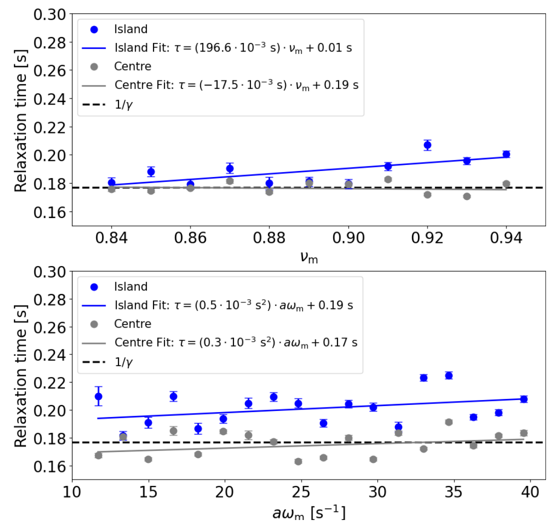

Figure 4: Relaxation time in the centre and island vs. ν_{m} (at constant amplitude) and $a\omega_{\mathrm{m}}$ together with fit functions.

CONCLUSIONS

The aim of the paper is to present a mathematical framework to describe non-linear beam manipulations in the presence of damping and noise effect using a model describing the longitudinal dynamics with modulation of the RF phase and close to a $1:1$ resonance condition. The properties of the equilibrium distributions close to the centre and in the stable island have been studied numerically as well as the relaxation time to equilibrium. The next step will be to extend this approach to understand the stochastic resonance trapping process when the modulation frequency and its amplitude are adiabatically changed.

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