MUELLER, SLATER & CASIMIR VERSUS KAHAN & PAPAS TIME TO SET THE RECORD STRAIGHT

S. Koscielniak*, TRIUMF, Vancouver, Canada

Abstract

The relative fractional change of resonance frequency equals the fractional change in stored energy. This relation predicts the change of EM frequency when the walls of a metallic cavity are deformed or small objects inserted. J.M. Müller derived this formula in 1939 by conceptual and mathematical errors. Casimir corrected those errors in 1949. The formula is also associated with J.C. Slater, published in 1946. Similar formulae were derived by Kahan in 1946 and by Papas in 1954. The Kahan version was reported, influentially, by Borgnis & Papas in the 1958 Handbook of Physics. Nevertheless, the relation is often called Slater's theorem. The Müller, Slater and Casimir versions rely only on the properties of eigenfunctions, and electromagnetic boundary conditions at a metallic surface. The Kahan and Papas short cut to the formula relies on thermodynamics, the adiabatic theorem, perfect thermal isolation and infinite time, and that the cavity contains EM oscillations in quadrature. The latter derivation (K-S) relies on assumptions that are artificial and unnecessary, and fails completely if there is no EM field present while the cavity is deformed. The situation of multiple derivations and attributions has led to some confusion. The time to set the record straight is overdue!

INTRODUCTION

The theory of electromagnetic (EM) waves in resonant cavities is not a modern subject. It was a short step, made by Lord Rayleigh pre-1900, from Maxwell's equations to studying trapped waves in closed cavities. And shortly afterward to the realization of the ultra-violet catastrophe (for cavity radiation) that spurred the early quantum-mechanics revolution post-1900. However, the modern understanding of EM cavity resonators in terms of eigenmodes each of which can be represented by an equivalent lumped-element (parallel) LCR-circuit stems from the period immediately proceeding and, in particular, during World War II. The importance of radio-communications, and RADAR, spurred many developments; including the theoretical basis for cavity resonance tuning and methods for measuring the EM field profile.

Frequency Tuning

The relative fractional change of resonance frequency (f) equals the fractional change in stored energy (W):

$$\delta f/f \;=\; \delta W/W\,. \tag{1}$$

Equation (1) is a consequence of Slater's theorem. Slater never wrote the relation in precisely this form. Equation (1) predicts the change of EM frequency when the walls of a metallic cavity are deformed (δV) or small objects (beads) inserted. The formulae for δV or beads are similar but distinct. This article concerns five separate versions of the derivation of Eq. (1). Three start from Maxwell's curl equations, use the vector identity

$$\nabla\cdot(\mathbf{E}\times\mathbf{H}) = \mathbf{H}\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot(\nabla\times\mathbf{H})\,, \tag{2}$$

and stipulate *perturbation at constant* electric and magnetic field $\mathbf{E}$, $\mathbf{H}$. A fourth/fifth derivation is loosely based on the adiabatic theorem for slow variations performed on a lossless Hamiltonian system. We shall take them in historical order.

IN ESSENCE

With hindsight, the Müller and Casimir derivations can be reduced to their essence. Take time-harmonic fields $e^{+i\omega t}$ with spatial parts $\mathbf{E}(\mathbf{x}), \mathbf{H}(\mathbf{x})$. Maxwell's curl equations:

$$A:\;\; \nabla\times\mathbf{E} + i\omega\mu\mathbf{H} = 0, \quad B:\;\; \nabla\times\mathbf{H} - i\omega\epsilon\mathbf{E} = 0\,. \tag{3}$$

For Müller take $\mathbf{H}^* A \; + \mathbf{E}^* B$ thus:

$$i\omega[\mu\mathbf{H}\mathbf{H}^* - \epsilon\mathbf{E}\mathbf{E}^*] + \mathbf{H}^*(\nabla\times\mathbf{E}) + \mathbf{E}^*(\nabla\times\mathbf{H}) = 0\,. \tag{4}$$

For Casimir take $\mathbf{H}^* A \; - \mathbf{E}^* B$ thus:

$$i\omega[\mu\mathbf{H}\mathbf{H}^* + \epsilon\mathbf{E}\mathbf{E}^*] + \mathbf{H}^*(\nabla\times\mathbf{E}) - \mathbf{E}^*(\nabla\times\mathbf{H}) = 0\,. \tag{5}$$

Here * denotes complex conjugate. For a lossless cavity resonator, EM fields are in quadrature. Hence $\mathbf{E}$ is real, and $\mathbf{H} \rightarrow -i\mathbf{H}$ with $\mathbf{H}$ real. We substitute Eq. (2) in Eq. (4). Let $\chi \equiv [\mathbf{H}(\nabla\times\mathbf{E}) + \mathbf{E}(\nabla\times\mathbf{H})]$. Let W_E and W_H be the electric and magnetic stored energy, respectively. We form the integrals of Eqs. (4, 5) over the cavity volume:

$$\omega[W_H - W_E] - \int_S (\mathbf{E}\times\mathbf{H})dS \;=\; 0\,, \tag{6}$$

$$\omega[W_H + W_E] - \int_V \chi(\mathbf{E},\mathbf{H})dV \;=\; 0\,. \tag{7}$$

Equation (6), Poynting's theorem [1] for a cavity, was used by Müller. In the unperturbed state, both terms in Eq. (6) are zero. Equation (7) was used by Casimir. Note that Eq. (7)+Eq. (6), and Eq. (7)-Eq. (6), gives expressions for W_H and W_E, respectively.

The next step is to form the total derivative of Eqs. (6) or (7) including variation of the surface and volume, while holding $\mathbf{E}, \mathbf{H}$ constant inside the volume (but not necessarily on the surface). Starting from Eq. (7) take plus sign in Eq. (8). Starting from Eq. (6) take the minus sign in Eq. (8).

$$\delta\omega[W_H \pm W_E] + \omega\int_{\Delta V}[H^2\delta\mu \pm E^2\delta\epsilon]dV \;+\; \omega\int_{\delta V}[\mu H^2 \pm \epsilon E^2]dV \;=\; J_\pm \tag{8}$$

* shane@triumf.ca

$$J_+ \equiv \int_{\delta V} \chi(\mathbf{E},\mathbf{H})dV, \qquad J_- \equiv \int_{\delta S} (\mathbf{E}\times\mathbf{H})dS. \quad (9)$$

If the cavity is filled with dielectric or ferrite then $\Delta V = V$; otherwise ΔV is the volume of the beads. Casimir used the eigenmode expansion to simplify the integral J_+.

MÜLLER

Müller's 1939 paper [2] is the basis for the so-called "bead-pull" measurements in which a dielectric or metallic bead (supported on a filament) is moved through a cavity resonator to find the "field profile", the spatial variation of the field modulus, in terms of a shift of the resonance frequency. Although the paper contains several good ideas, there are serious mathematical flaws. Müller makes no use of the eigenmode expansion. Müller begins with the complex-algebra form [3] of Poynting's theorem for time-harmonic EM fields and a closed, perfectly conducting shell. He then considers perturbations. Unfortunately, Eq. (1) cannot be derived from Poynting's theorem. Because the theorem is about the *rate of change* of energy, the constant term $(W_H + W_E)$ must be absent. Müller replaces (without justification) the minus signs in Eqs. (4, 6) by + signs. Nevertheless, the results derived by Müller from Eq. (1) concerning bead-pulls are correct; leading the work to be cited by Bethe [4] and Schwinger (1943), Hendrik Casimir [5] (1951), Handbuch der Physik (1958), and Dieter Schulze [6] (1971), etc.

The paper divides into two parts: before and after Müller's Eq. (8) in the original manuscript [2] which is Eq. (10) here:

$$\delta\omega[W_H + W_E] + \omega\int_{\Delta V}[H^2\delta\mu + E^2\delta\epsilon]dV \;+\; \omega\int_{\delta V}[\mu H^2 - \epsilon E^2]dV \;=\; J_- \quad (10)$$

Müller's Eq. (8) relates a relative fractional shift in frequency $\delta\omega/\omega$ to a fractional change in stored energy. The first part, Secs. 1-4, appear to establish the basis for Eq. (8). The second part, Secs. 5-7, apply his Eq. (8) to a small perturbing dielectric or metallic sphere to find the frequency shift. Curiously, while the second part is confident and error free, the first part is stumbling and filled with mathematical errors. Ultimately, the key Eq. (8) cannot be derived from Poynting's theorem for EM cavity fields. Müller indicates a more general basis for his Eq. (8) in his Eq. (9) (Eq. (1) in this paper) which proposes the frequency shift is proportional to the work (δW) performed on the system. This later relation is essentially the adiabatic theorem. Perhaps, Secs. 1-4 were really motivated by the (failed) intuition that a relation like Eq. (1) could be established for EM-fields using the Poynting theorem. In contrast, his Secs. 5-7 dealing with "beads" are essentially correct, experimentally valuable, and need no commentary here.

SLATER

John Slater worked at the MIT Radiation laboratory (during WWII) on the development of microwave techniques. Immediately after the war, much of the MIT work was declassified. In 1946 Slater's summary of the theoretical and practical advances made, "Microwave Electronics", was published in Physical Review [7]. The material was later greatly extended and published as a book [8] in 1950. However, Chap. 4 of Ref. [8] is essentially the same material as in Ref. [7]. The theory of cavity modes, and expansion in terms of orthogonal functions, is prominent but far from being the only topic in this summary. Slater notes other authors (E. Condon and J. Schwinger) working independently on mode expansion.

Section III-1 presents the orthogonal functions for the hollow cavity, and their vector-calculus properties. Section III-2 presents Maxwell's equations for the hollow cavity in terms of expansions in normal modes. $\nabla\times\mathbf{E} + \partial\mathbf{B}/\partial t = \mathbf{0}$ leads to his Eq. (III.34). $\nabla\times\mathbf{H} - \partial\mathbf{D}/\partial t = \mathbf{J}$ leads to his Eq. (III.35). The equations are combined to give second-order differential equations, Eqs. (III.42) and (III.43) for $\mathbf{E}$ and $\mathbf{H}$ respectively. *Slater's Theorem* is Eqs. (III.34, 35) or Eqs. (III.42, 43). In Sec. III-3 to III-5 he introduces ohmic losses and makes the formal analogy of the EM mode to a driven LCR oscillator. Section III-7 presents the effect of perturbing the metallic boundaries leading to a volume change δV. The vector identity Eq. (2) is used to transform a surface integral leading to

$$\omega^2 = \omega_a^2\left[1 + \int_{\delta V}(H_a^2 - E_a^2)dV\right], \;\; \text{mode index } a\,. \quad (11)$$

This (Eq. (III.89) in the original) has been called Slater's perturbation theorem. The eigenfunctions E_a, H_a are orthonormalized, so there is no need to divide $\int_{\delta V}\dots$ by W. The frequency shift due to conducting or dielectric beads is given in Eqs. (III.82-84). Slater's theorem is cited by Yamazaki [9].

ADIABATIC INVARIANT

Let Q, P be conjugate Hamiltonian coordinates, e.g. position and momentum or energy and time. For an oscillator system, periodically cycling, the path integral

$$I = \oint P dQ = \langle H/\omega\rangle \;\; \text{is an adiabatic invariant} \;\; \delta I = 0\,.$$

H is the Hamiltonian, and ω the angular oscillation frequency. The invariant is attributed to Boltzmann [10]. The original intent, going back to Rayleigh [11], of the adiabatic theorem was to predict the change in amplitude in response to slow variation of system parameters that determine the oscillation frequency, for example changing the steepness of a confining potential well. The amplitudes Q, P of the motion (and H, ω) may vary widely, but I is (almost) constant. Ehrenfest gave a quantum interpretation in 1911.

KAHAN

In 1946, Théodore Kahan presented [12] an entirely different derivation of Eq. (1). He begins from Boltzmann's theorem [10] for the incremental heat (δQ) per cycle period T: $\delta Q = (2T)\delta(T\times K)$ where K is the kinetic energy. For certain systems $K = W/2$ where W is the total energy (kinetic plus potential). Hence $\delta Q = \delta(TW)/T$. For a reversible variation (no increase of entropy) $\delta Q = 0$, leading to the adiabatic invariant $\delta(T.W) = 0$ or $\delta W - (W/\omega)\delta\omega = 0$. Without further explanation, Kahan identifies the total energy with $W = \int_V [\epsilon E^2 + \mu H^2] dV$. And hence for small perturbations:

$$\begin{aligned}\delta W &= \int_V [\mu \delta H^2 + \epsilon \delta E^2] dV \\ &+ \int_{\Delta V} [H^2 \delta\mu + E^2 \delta\epsilon] dV + \int_{\delta V} [\mu H^2 + \epsilon E^2] dV .\end{aligned} \quad (12)$$

Let us be clear the assumptions being made:

- the properties of EM cavity normal modes can be taken for granted,
- the fields E, H are present at all times,
- the variations are made infinitely slowly,
- the system is Hamiltonian (perfectly lossless),
- there is no external drive.

The latter assumptions imply that the formalism is incompatible with the perturbative calculation of losses or quality factor. Note: Kahan does not give explicit formulae for the perturbations due to small metallic or dielectric objects. Note also the + sign in the integral $\int_{\delta V} \dots$ (Eq. (12)) is incorrect for the work done against radiation pressure. A few months later, Kahan applied [13] his theory to radiation pressure; specifically to a piston or diaphragm in a cylindrical cavity. The sign error is not corrected.

CASIMIR

Hendrik Casimir gave a series of lectures on EM waves in cavity resonators in 1949. The lectures, published [5] in 1951, lead from basic principles to the perturbation theory for beads. Although he cites Müller, Casimir generously does not mention the mathematical errors; and instead provides his own derivation (using the Slater eigenmode expansion) of the perturbation formulae Eq. (8) for metallic, dielectric and ferrite spheres. The last enables to find the permeability tensor.

PAPAS

In 1954 Charles Papas wrote a brief paper [14] which is the antecedent of the 1958 entry [15] in Handbuch der Physik. Here Papas acknowledges Müller and Slater as the independent authors of Eq. (1). Unaware of Kahan, Papas then proposes a *most simple derivation of the M-S formula based on thermodynamics* (i.e. the adiabatic theorem) for the frequency shift due to small beads.

BORGNIS AND PAPAS

Borgnis [15] and Papas gave an influential account (Sec. G, Articles 31-33) of the properties of cavity resonators in the august Encyclopedia of Physics (1958). They begin in Article 32 *Small perturbations*, with the complex Poynting theorem and Müllers purported derivation of Eq. (1). However, they do not to reproduce Müller's derivation, choosing instead to outline the approach of Kahan [12, 13]. Borgnis and Papas state a version of the Boltzmann-Ehrenfest theorem W/ω = constant; and then recover results similar to Kahan. They correct the sign error for radiation pressure in Kahan's integral $\int_{\delta V} \dots$. In a departure from Kahan, Borgnis and Papas give *explicit* formulae for the perturbations due to small metallic or dielectric or ferrite spheres, as are needed in bead-pull measurements or in the measurement of material properties. They also give an explicit formula for Lorentz-force detuning, again based on Kahan's notion of work performed on the system infinitely slowly. In Article 33, "Forced Oscillations", despite extensive use of the eigenmode expansion, Borgnis and Papas make no mention of the work of Slater.

Unfortunate Consequence

The Borgnis-Papas entry has the unfortunate consequence of cementing the Kahan thermodynamic derivation on the same (or even superior) footing as the Maxwell-eigenmode derivations of Müller, Slater and Casimir (M-S-C). *Let us be clear what Kahan's result cannot do.* It does not predict the resonance frequency shift when we take a cold cavity (no EM field), strike it with a mallet to deform the shell, and then apply an RF drive. In contrast, the MSC derivation predicts the new frequency provided only that we know the local volume change of the metallic shell and the mode.

The Kahan derivation mis-interprets the Boltzmann adiabatic theorem in two ways: (1) The Boltzmann relation is only for a dynamical system in a state of continuing oscillation. (2) Kahan has inverted cause and effect: actually the changes in oscillation amplitude (effect) are due to the changes in oscillation frequency (cause).

CONFUSION REIGNS

The situation of multiple derivations and attributions has led to some confusion. For example, Whittun [16, 17] calls Eq. (1) Slater's formula but recounts the Kahan-Papas-Borgnis derivation, as does Delayen [18, 19]. Both Whittun and Delayen work up a quasi-quantum version of the adiabatic theorem based on a photon gas of a fixed number of photons equal to $W/(\hbar\omega)$. Contrastingly, Apel [20] recounts Casimir's derivation but attributes the result to Müller; and Schulze [6] relies on the derivation of Borgnis/Papas, but makes attribution to Müller.

CONCLUSION

This brief history chronicles the various contributors and authors of what is often called Slater's theoerm. An English translation of Müller's paper, and a commentary, is given in Ref. [21]. We recommend the name Müller-Slater-Casimir theorem; and for the purpose of text books presenting the Slater version starting from Eqs. (III.42, 43).

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REFERENCES

[1] J. H. Poynting, "On the transfer of energy in the electromagnetic field", *Philos. Trans. R. Soc. Lond.*, vol. 175, pp. 343–361, 1884. doi:10.1098/rstl.1884.0016

[2] J. Müller, "Untersuchungen über elektromagnetische Hohlräume", *Hochfrequenztechnik und Elektroakustik*, vol. 54, pp. 157–161, 1939.

[3] F. Emde, "Die komplexe Rechnung bei Schwingungen", *Elektrotech. Maschinenbau*, vol. 27, pp. 111–115, 1909.

[4] H. A. Bethe and J. Schwinger, National Defense Research Committee Report No. D1-117, Cornell University, 1943, unpublished.

[5] H. B. G. Casimir, "On the theory of electromagnetic waves in resonant cavities", *Philips Res. Rep.*, vol. 6, pp. 162–182, 1951.

[6] D. Schulze, "Ponderomotive Stability of RF Resonators and Resonator Control Systems", Kernforschungszentrum Karlsruhe, Karlsruhe, Germany, Rep. KFK-1493, 1971; translation: Argonne National Laboratory, Lemont, IL, USA, Rep. ANL-TRANS-944. doi:10.2172/4659728

[7] J. C. Slater, "Microwave electronics", *Rev. Mod. Phys.*, vol. 18, no. 4, pp. 441–512, Oct. 1946. doi:10.1103/RevModPhys.18.441

[8] J. C. Slater, *Microwave Electronics*. Boston, MA, USA: D. Van Nostrand, 1950.

[9] Y. Yamazaki, "Microwave electronics – Slater's perturbation theorem", in *RF Engineering for Particle Accelerators*, Proc. JAPAN-CERN-US Joint Accelerator School, Japan, 1996. Singapore: World Scientific, 1999.

[10] L. Boltzmann, *Vorlesungen über die Prinzipien der Mechanik*, vol. 2. Leipzig, Germany: J. A. Barth, 1904.

[11] Lord Rayleigh, "On the pressure of vibrations", *Philos. Mag.*, vol. 3, no. 15, pp. 338–346, 1902. doi:10.1080/14786440209462769

[12] T. Kahan, "The law of slow transformation (of Boltzmann) and the theory of electromagnetic cavities", *C. R. Acad. Sci. Paris*, vol. 222, p. 70, 1946.

[13] T. Kahan, "Perturbations and radiation pressure in electromagnetic cavities", *C. R. Acad. Sci. Paris*, vol. 223, p. 785, 1946.

[14] C. H. Papas, "Thermodynamic consideration of electromagnetic cavity resonators", *J. Appl. Phys.*, vol. 25, pp. 1552–1553, 1954. doi:10.1063/1.1721592

[15] F. E. Borgnis and C. H. Papas, "Electromagnetic waveguides and resonators", in *Handbuch der Physik*, vol. XVI, S. Flügge, Ed. Berlin, Germany: Springer-Verlag, 1958, pp. 406–415. doi:10.1007/978-3-642-45910-3_8

[16] D. H. Whittum, "Introduction to electrodynamics for microwave linear accelerators", SLAC National Accelerator Laboratory, Menlo Park, CA, USA, Rep. SLAC-PUB-7802, 1998.

[17] D. H. Whittum, "Introduction to microwave linacs", SLAC National Accelerator Laboratory, Menlo Park, CA, USA, Rep. SLAC-PUB-8026, 1998.

[18] J. R. Delayen, "Ponderomotive instabilities and microphonics – a tutorial", in *Proc. 12th Int. Workshop on RF Superconductivity (SRF'05)*, Ithaca, NY, USA, Jul. 2005, p. 35.

[19] J. R. Delayen, "Ponderomotive instabilities and microphonics – a tutorial", *Physica C*, vol. 441, pp. 1–6, 2006. doi:10.1016/j.physc.2006.03.050

[20] R. Apel, "Entwurf und Realisierung einer planaren Stehwellenstruktur für Millimeterwellen", Ph.D. thesis, Technische Universität Berlin, Berlin, Germany, 2003.

[21] S. R. Koscielniak, "Poynting's theorem and derived results", TRIUMF, Vancouver, BC, Canada, Rep. TRI-BN-26-05, Apr. 2026.