UNIVERSAL REPRESENTATION OF CHROMATIC ABERRATION IN ELECTRON BEAM OPTICS

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Abstract

We present a novel unified approach to the expression and correction of electron beam chromatic aberrations, which comprehensively covers optical systems of both a single-pass beam transport line and a circular accelerator satisfying periodic boundary conditions. In an electron circular accelerator, linear chromaticity has been utilized to express the first-order difference in beam focusing caused by the electron energy deviation, which can be computed by a well-known formula using a beam envelope function. However, this conventional formulation is incompatible with non-periodic beam transport lines, such as XFELs driven by linacs and the final focus system of linear colliders. We address this discrepancy by deriving a generalized formula that seamlessly connects the two different accelerator topologies. Our representation not only reproduces the well-known ring chromaticity but also provides a consistent and practical definition of chromatic aberrations for a single-pass transport line and their correction methods. This presentation outlines one approach to resolving this apparent contradiction and the universal formula to express chromatic effects.

INTRODUCTION

There are two main types of accelerator topologies. One is a "closed" transport system that satisfies periodic boundary conditions, and the other is an "open" transport system without any conditions. A typical example of the former is a storage ring, where machine parameters such as Courant-Snyder or Twiss parameters [1, 2] are automatically determined to satisfy the boundary conditions. An example of the latter is a beam transport line or a single-pass linac such as linear colliders or XFEL linacs, and machine parameters are obtained by tracing the initial values at the entrance to the downstream point. In both "open" and "closed" transport systems, the correction of chromatic aberration is an important topic, and for a better understanding and correction, a general formulation that seamlessly describes the chromatic aberration for the two different accelerator topologies will be helpful.

As for the linear chromaticity in the open transport system, it is calculated as the first-order term of the energy-dependent component of the betatron phase traced from the entrance [3]. In contrast, in the closed transport system such as storage rings, a well-known formula is used for calculating the linear chromaticity [2]. Thus, even when considering the chromaticity alone, it does not appear to be straightforward to describe the two accelerator topologies in a unified manner.

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In this work, we present a unified formulation for describing the chromatic aberration of both open and closed beam transport systems [4]. Since the betatron motion can be described by the linear equation of motion with quadrupole field strengths multiplied by a factor $1/(1+\delta)$, where δ is the relative momentum deviation, the transfer matrix formalism is applicable [5-8]. With this in mind, we adopt a chromatic aberration-decoupled formalism, which separately treats ideal particle motion with a design momentum ($\delta = 0$) and perturbative motion due to non-zero δ. With this formulation, we treat the betatron motion up to the first order of δ, and we can derive a formula of chromatic aberration seamlessly connecting the two different accelerator topologies. Boundary conditions can also be taken into account easily. As an example of the application of our formulation, we show an efficient correction scheme of the chromatic aberration for a tentative chicane lattice designed for the BL3 beamline of SPring-8 Angstrom Compact free electron LAser (SACLA) [9].

FORMULATION

The betatron motion for off-momentum particles in a beam transport line can be described by the transfer matrix formalism with quadrupole field strengths multiplied by a factor $1/(1+\delta)$, where $\delta \equiv \Delta p/p_0$ is the relative momentum deviation:

$$\begin{pmatrix} x^{[out]} \\ x'^{[out]} \end{pmatrix} = M \begin{pmatrix} x^{[in]} \\ x'^{[in]} \end{pmatrix}. \tag{1}$$

The transfer matrix M is parametrized as

$$M \equiv \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{\beta^{[out]}}{\beta^{[in]}}}(\cos\psi + \alpha^{[in]}\sin\psi) & \sqrt{\beta^{[in]}\beta^{[out]}}\sin\psi \\ \frac{\alpha^{[in]}-\alpha^{[out]}}{\sqrt{\beta^{[in]}\beta^{[out]}}}\cos\psi - \frac{1+\alpha^{[in]}\alpha^{[out]}}{\sqrt{\beta^{[in]}\beta^{[out]}}}\sin\psi & \sqrt{\frac{\beta^{[in]}}{\beta^{[out]}}}(\cos\psi - \alpha^{[out]}\sin\psi) \end{pmatrix}, \tag{2}$$

and the Courant-Snyder parameters (β, α, γ) is transformed as

$$\begin{pmatrix} \beta^{[out]} \\ \alpha^{[out]} \\ \gamma^{[out]} \end{pmatrix} = \begin{pmatrix} m_{11}^2 & -2m_{11}m_{12} & m_{12}^2 \\ -m_{11}m_{21} & 1+2m_{12}m_{21} & -m_{12}m_{22} \\ m_{21}^2 & -2m_{21}m_{22} & m_{22}^2 \end{pmatrix} \begin{pmatrix} \beta^{[in]} \\ \alpha^{[in]} \\ \gamma^{[in]} \end{pmatrix}. \tag{3}$$

We now expand M and (β, α, γ) in terms of δ so that we can decouple the terms due to chromatic aberration and separately treat ideal particle motion with a design momentum ($\delta = 0$) and perturbative motion with a non-zero δ. Up to the first order of δ, we write

$$M = M_0 + M_1\delta + O(\delta^2)\,, \tag{4}$$

$$\beta = \beta_0 + \beta_1\delta + O(\delta^2)\,, \tag{5}$$

$$\alpha = \alpha_0 + \alpha_1\delta + O(\delta^2)\,, \tag{6}$$

$$\gamma = \gamma_0 + \gamma_1\delta + O(\delta^2)\,, \tag{7}$$

and insert Eqs. (4)-(7) to Eq. (2) to obtain

$$\begin{pmatrix}\beta_0^{[out]}\\ \alpha_0^{[out]}\\ \gamma_0^{[out]}\\ \beta_1^{[out]}\\ \alpha_1^{[out]}\\ \gamma_1^{[out]}\end{pmatrix} = \begin{pmatrix}T & O\\ U & T\end{pmatrix}\begin{pmatrix}\beta_0^{[in]}\\ \alpha_0^{[in]}\\ \gamma_0^{[in]}\\ \beta_1^{[in]}\\ \alpha_1^{[in]}\\ \gamma_1^{[in]}\end{pmatrix}, \tag{8}$$

where

$$T = \begin{pmatrix} m_{0,11}^2 & -2m_{0,11}m_{0,12} & m_{0,12}^2\\ -m_{0,11}m_{0,21} & 1+2m_{0,12}m_{0,21} & -m_{0,12}m_{0,22}\\ m_{0,21}^2 & -2m_{0,21}m_{0,22} & m_{0,22}^2\end{pmatrix}, \tag{9}$$

$$U = \begin{pmatrix} 2m_{0,11}m_{1,11} & -2\left(m_{0,11}m_{1,12}+m_{1,11}m_{0,12}\right) & 2m_{0,12}m_{1,12}\\ -m_{0,11}m_{1,21}-m_{1,11}m_{0,21} & 2\left(m_{0,12}m_{1,21}+m_{1,12}m_{0,21}\right) & -m_{0,12}m_{1,22}-m_{1,12}m_{0,22}\\ 2m_{0,21}m_{1,21} & -2\left(m_{0,21}m_{1,22}+m_{1,21}m_{0,22}\right) & 2m_{0,22}m_{1,22}\end{pmatrix}, \tag{10}$$

and $m_{0,ij}$ ($m_{1,ij}$) is the *ij*-component of the matrix M_0 (M_1). The betatron phase is also expanded as

$$\psi = \psi_0 + \psi_1\delta + O(\delta^2)\,, \tag{11}$$

and using the following relation

$$\tan\psi = \frac{m_{12}}{\beta^{[in]}m_{11} - \alpha^{[in]}m_{12}}\,, \tag{12}$$

we have

$$\psi_1 = \frac{\tan^2\psi_0}{1+\tan^2\psi_0}\left\{\alpha_1^{[in]} - \frac{\beta_1^{[in]}m_{0,11}+\beta_0^{[in]}m_{1,11}}{m_{0,12}} + \frac{\beta_0^{[in]}m_{0,11}m_{1,12}}{\left(m_{0,12}\right)^2}\right\}. \tag{13}$$

This can be regarded as the linear chromaticity [3] for the "open" system such as a single-pass beam transport line:

$$\xi_{\text{OPEN}} \equiv \frac{\psi_1}{2\pi}\,. \tag{14}$$

ENERGY DEPENDENCE OF THE TRANSFER MATRIX

In this section we discuss the δ-dependent component of the transfer matrix M_1. The bending magnets are assumed to be of the sector type and can be treated approximately as a drift space, except that the dispersion function η_x is generated. The quadrupole (Q) and sextupole (S) magnets are treated using the thin-lens approximation, but the discussion does not lose its generality since a finite-length magnet can be divided into a sufficient number of segments and treated as a "kick-drift-kick..." sequence. For a sextupole magnet, the quadrupole field component is calculated using η_x at that point. Up to the first order of δ, the matrix M_1 can be written as (see Fig. 1):

$$M_1 = \textstyle\sum_{i\in Q,S} M_0^{[n,i]}V_1^{[i]}M_0^{[i,0]}\,, \tag{15}$$

where $M_0^{[j,i]}$ is the δ-independent component of the transfer matrix from the *i*-th kick position $s = s^{[i]}$ to the *j*-th kick position $s = s^{[j]}$ ($s = s^{[0]}$ at the entrance and $s = s^{[n]}$ at the exit), and $V_1^{[i]}$ is the δ-dependent component of the *i*-th kick:

$$V_1^{[i]} = \begin{pmatrix}0 & 0\\ k^{[i]} & 0\end{pmatrix}, \tag{16}$$

$$k^{[i]} \equiv \begin{cases} \pm\,(K_0L)^{[i]}:\ i\in \mathrm{Q}\\ \mp\,(\Lambda_0L)^{[i]}\eta_x^{[i]}:\ i\in \mathrm{S}\end{cases}, \tag{17}$$

with (K_0L) and (Λ_0L) being the integrated field strength of the quadrupole and sextupole magnet, respectively.

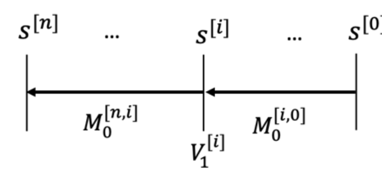

Figure 1: The first-order transfer matrix for δ.

Using Eq. (16) in Eq. (15), M_1 can be expressed as follows:

$$M_1 = \begin{pmatrix} \sqrt{\dfrac{\beta_0^{[n]}}{\beta_0^{[0]}}}\left(F_{SC}+\alpha_0^{[0]}F_{SS}\right) & \sqrt{\beta_0^{[0]}\beta_0^{[n]}}F_{SS}\\ \dfrac{1}{\sqrt{\beta_0^{[0]}\beta_0^{[n]}}}\left(F_{CC}+\alpha_0^{[0]}F_{CS}-\alpha_0^{[n]}F_{SC}-\alpha_0^{[0]}\alpha_0^{[n]}F_{SS}\right) & \sqrt{\dfrac{\beta_0^{[0]}}{\beta_0^{[n]}}}\left(F_{CS}-\alpha_0^{[n]}F_{SS}\right)\end{pmatrix}, \tag{18}$$

where

$$F_{SS} \equiv \textstyle\sum_{i\in Q,\,S} k^{[i]}\beta_0^{[i]}\sin\psi_0^{[n,i]}\,\sin\psi_0^{[i,0]} = \sin\psi_0\,W_{SC} - \cos\psi_0\,W_{SS}\,, \tag{19}$$

$$F_{SC} \equiv \textstyle\sum_{i\in Q,\,S} k^{[i]}\beta_0^{[i]}\sin\psi_0^{[n,i]}\,\cos\psi_0^{[i,0]} = \sin\psi_0\,W_{CC} - \cos\psi_0\,W_{SC}\,, \tag{20}$$

$$F_{CS} \equiv \textstyle\sum_{i\in Q,\,S} k^{[i]}\beta_0^{[i]}\cos\psi_0^{[n,i]}\,\sin\psi_0^{[i,0]} = \cos\psi_0\,W_{SC} + \sin\psi_0\,W_{SS}\,, \tag{21}$$

$$F_{CC} \equiv \sum_{i\in Q,S} k^{[i]}\beta_0^{[i]} \cos\psi_0^{[n,i]} \cos\psi_0^{[i,0]}$$
$$= \cos\psi_0 W_{CC} + \sin\psi_0 W_{SC}, \quad (22)$$

with

$$W_{SS} \equiv \sum_{i\in Q,S} k^{[i]}\beta_0^{[i]} \sin^2\psi_0^{[i,0]}, \quad (23)$$

$$W_{CC} \equiv \sum_{i\in Q,S} k^{[i]}\beta_0^{[i]} \cos^2\psi_0^{[i,0]}, \quad (24)$$

$$W_{SC} \equiv \sum_{i\in Q,S} k^{[i]}\beta_0^{[i]} \sin\psi_0^{[i,0]} \cos\psi_0^{[i,0]}. \quad (25)$$

The betatron phase between $s^{[i]}$ and $s^{[j]}$ is denoted as $\psi_0^{[j,i]}$, and $\psi_0 \equiv \psi_0^{[n,0]}$. The chromaticity defined by Eq. (14) is calculated by

$$\psi_1 = \frac{\tan^2\psi_0}{1+\tan^2\psi_0}$$
$$\times\left\{\alpha_1^{[0]} - \frac{\beta_1^{[0]}m_{0,11}+\beta_0^{[0]}m_{1,11}}{m_{0,12}} + \frac{\beta_0^{[0]}m_{0,11}m_{1,12}}{(m_{0,12})^2}\right\}, \quad (26)$$

$$\tan\psi_0 = \frac{m_{0,12}}{\beta_0^{[0]}m_{0,11}-\alpha_0^{[0]}m_{0,12}}. \quad (27)$$

APPLICATION 1: CHROMATICITY UNDER PERIODIC BOUNDARY CONDITIONS

When periodic boundary conditions are imposed at the entrance and at the exit, the system can be regarded as "closed" like a storage ring. In this case, the chromaticity defined by Eq. (14) must be reduced to the following well-known formula:

$$\xi_{\text{CLOSED}} = -\frac{1}{4\pi}\sum_{i\in Q,S} k^{[i]}\beta_0^{[i]}. \quad (28)$$

We now show this by using the formulation developed in the previous section. When the periodic boundary conditions are imposed at both ends of the transport system, the Courant-Snyder parameters satisfy the following:

$$\beta_0^{[n]} = \beta_0^{[0]} \equiv \beta_0, \alpha_0^{[n]} = \alpha_0^{[0]} \equiv \alpha_0, \gamma_0^{[n]} = \gamma_0^{[0]} \equiv \gamma_0, \quad (29)$$

$$\beta_1^{[n]} = \beta_1^{[0]} \equiv \beta_1, \alpha_1^{[n]} = \alpha_1^{[0]} \equiv \alpha_1, \gamma_1^{[n]} = \gamma_1^{[0]} \equiv \gamma_1, \quad (30)$$

where

$$\gamma_0 = \frac{1+\alpha_0^2}{\beta_0}, \quad \gamma_1 = \frac{2\alpha_0\alpha_1-\gamma_0\beta_1}{\beta_0}. \quad (31)$$

The relations (31) are derived by substituting Eqs. (5)-(7) into $\gamma = (1+\alpha^2)/\beta$ and comparing the coefficients of the powers of δ [8]. Under these conditions, Eq. (8) is written as

$$\begin{pmatrix}\beta_1\\ \alpha_1\\ \gamma_1\end{pmatrix} = U\begin{pmatrix}\beta_0\\ \alpha_0\\ \gamma_0\end{pmatrix} + T\begin{pmatrix}\beta_1\\ \alpha_1\\ \gamma_1\end{pmatrix}, \quad (32)$$

and this can be solved for β_1 and α_1, yielding the following periodic solution [4]:

$$\beta_1 = \frac{\beta_0}{2\tan\psi_0}(W_{CC} - W_{SS}) + \beta_0 W_{SC}, \quad (33)$$

$$\alpha_1 = \frac{1}{2}(W_{CC} - W_{SS} + 2\alpha_0 W_{SC})$$
$$+\frac{1}{2\tan\psi_0}\{\alpha_0(W_{CC} - W_{SS}) - 2W_{SC}\}. \quad (34)$$

Equation (26) is reduced to

$$\psi_1 = \alpha_1\sin^2\psi_0 - \frac{\beta_1}{\beta_0}(\cos\psi_0 + \alpha_0\sin\psi_0)\sin\psi_0$$
$$-\sin\psi_0 m_{1,11} + \frac{1}{\beta_0}(\cos\psi_0 + \alpha_0\sin\psi_0)m_{1,12}, \quad (35)$$

and by using Eqs. (33) and (34), we finally obtain

$$\psi_1 = -\frac{1}{2}(W_{CC} + W_{SS}). \quad (36)$$

From Eqs. (23) and (24), we see that $\psi_1/(2\pi)$ agrees with Eq. (28).

APPLICATION 2: CORRECTION OF CHROMATIC ABERRATION FOR LATTICE WITH MIRROR-SYMMETRY

We next consider the case where the excitation pattern of quadrupole and sextupole magnets is mirror-symmetric with respect to the center of the beam transport system and the δ-independent component of the betatron function β_0 is also mirror-symmetric:

$$\beta_0^{[n]} = \beta_0^{[0]}, \alpha_0^{[n]} = -\alpha_0^{[0]}, \gamma_0^{[n]} = \gamma_0^{[0]}. \quad (37)$$

For the mirror-symmetric configuration, we can show that

$$F_{CS} = F_{SC} = \frac{1}{2}\sin\psi_0 (W_{SS} + W_{CC}), \quad (38)$$

and Eq. (18) is reduced to

$$M_1 = (W_{SS} + W_{CC})\begin{pmatrix}\frac{1}{2}\sin\psi_0 & 0\\ \frac{1}{\beta_0^{[0]}}\left(\cos\psi_0 + \alpha_0^{[0]}\sin\psi_0\right) & \frac{1}{2}\sin\psi_0\end{pmatrix}$$
$$+F_{SS}\begin{pmatrix}\alpha_0^{[0]} & \beta_0^{[0]}\\ \gamma_0^{[0]} & \alpha_0^{[0]}\end{pmatrix}. \quad (39)$$

When the δ-dependent components of the Courant-Snyder parameters are negligibly small at the entrance of the transport system, we can put

$$\beta_1^{[0]} = 0, \alpha_1^{[0]} = 0, \gamma_1^{[0]} = 0, \quad (40)$$

and by using Eqs. (26) and (31), we have

$$\xi_{\text{OPEN}} = \frac{-1}{4\pi}\{\sin^2\psi_0 (W_{SS} + W_{CC}) - 2\cos\psi_0 F_{SS}\}. \quad (41)$$

The chromatic amplitude function $\widetilde{W}$ [10] at the exit of the transport system is defined as

$$\widetilde{W} = \sqrt{\left(\alpha_1^{[n]} - \frac{\alpha_0^{[n]}\beta_1^{[n]}}{\beta_0^{[n]}}\right)^2 + \left(\frac{\beta_1^{[n]}}{\beta_0^{[n]}}\right)^2}. \quad (42)$$

Under the conditions (40), Eq. (8) can be written as

$$\begin{pmatrix} \beta_1^{[n]} \\ \alpha_1^{[n]} \\ \gamma_1^{[n]} \end{pmatrix} = U \begin{pmatrix} \beta_0^{[0]} \\ \alpha_0^{[0]} \\ \gamma_0^{[0]} \end{pmatrix}, \tag{43}$$

where U is given by Eq. (10). From Eq. (43), we obtain

$$\beta_1^{[n]} = \beta_0^{[0]} \sin\psi_0 \{(W_{SS} + W_{CC})\cos\psi_0 + 2F_{SS}\}, \tag{44}$$

$$\alpha_1^{[n]} = -\left(\cos\psi_0 + \alpha_0^{[0]} \sin\psi_0\right) \times \{(W_{SS} + W_{CC})\cos\psi_0 + 2F_{SS}\}, \tag{45}$$

and Eq. (42) is reduced to

$$\widetilde{W} = |(W_{SS} + W_{CC})\cos\psi_0 + 2F_{SS}|. \tag{46}$$

We now consider the correction of chromatic aberration. If we could set

$$M_1 = O, \tag{47}$$

by tuning the lattice (e.g., sextupole magnet strengths), the matrix U on the right-hand side of Eq. (43) would vanish. This means that the δ-dependent component of the Courant-Snyder parameters at the exit $\left(\beta_1^{[n]}, \alpha_1^{[n]}, \gamma_1^{[n]}\right)$ vanishes regardless of the initial values at the entrance $\left(\beta_0^{[0]}, \alpha_0^{[0]}, \gamma_0^{[0]}\right)$, and this is a desirable property for transport systems like FEL beamlines, as it ensures that fluctuations of the beam envelope do not affect subsequent stages. However, in practice, it is not easy to realize such a lattice that strictly satisfies the condition $M_1 = O$, since dedicated tuning knobs are required and they tend to be strong. We then look for an alternative and effective way of correcting chromatic aberration by using a smaller number of tuning knobs with appropriate field strengths.

The most natural way to correct the chromatic aberration will be to apply a chromaticity correction, since the energy dependence of the betatron phase is eliminated on average and the conditions (47) will be approximately satisfied.

Correction Scheme (A)

If we require

$$\xi_{\mathrm{CLOSED}} = 0, \tag{48}$$

we have

$$W_{SS} + W_{CC} = 0, \tag{49}$$

from Eqs. (28) and (36). The matrix M_1 is written as

$$M_1 = F_{SS} \begin{pmatrix} \alpha_0^{[0]} & \beta_0^{[0]} \\ \gamma_0^{[0]} & \alpha_0^{[0]} \end{pmatrix}. \tag{50}$$

The right-hand side represents the residuals of M_1 in the present correction scheme, and if F_{SS} is suppressed to a small number, the condition (47) is approximately satisfied.

Correction Scheme (B)

As a slight variation, we can require

$$\xi_{\mathrm{OPEN}} = 0, \tag{51}$$

instead of Eq. (48). In this case, from Eqs. (41) and (39), we have

$$M_1 = (W_{SS} + W_{CC}) \times \left\{ \begin{pmatrix} \frac{1}{2}\sin\psi_0 & 0 \\ \frac{1}{\beta_0^{[0]}}\left(\cos\psi_0 + \alpha_0^{[0]}\sin\psi_0\right) & \frac{1}{2}\sin\psi_0 \end{pmatrix} + \frac{1}{2}\sin\psi_0 \tan\psi_0 \begin{pmatrix} \alpha_0^{[0]} & \beta_0^{[0]} \\ \gamma_0^{[0]} & \alpha_0^{[0]} \end{pmatrix} \right\}. \tag{52}$$

Correction Scheme (C)

Another correction scheme will be to suppress the chromatic amplitude function $\widetilde{W}$ given by Eq. (46):

$$\widetilde{W} = 0. \tag{53}$$

We then have

$$M_1 = (W_{SS} + W_{CC}) \times \left\{ \begin{pmatrix} \frac{1}{2}\sin\psi_0 & 0 \\ \frac{1}{\beta_0^{[0]}}\left(\cos\psi_0 + \alpha_0^{[0]}\sin\psi_0\right) & \frac{1}{2}\sin\psi_0 \end{pmatrix} - \frac{1}{2}\cos\psi_0 \begin{pmatrix} \alpha_0^{[0]} & \beta_0^{[0]} \\ \gamma_0^{[0]} & \alpha_0^{[0]} \end{pmatrix} \right\}. \tag{54}$$

Numerical Example

To check and compare the effectiveness of the above correction schemes, we apply them to an example lattice shown in Fig. 2, which has been designed tentatively for upgrading the BL3 beamline of SPring-8 Angstrom Compact free electron LAser (SACLA) [9]. This lattice incorporates a chicane with a positive longitudinal dispersion R_{56}, enabling bunch compression to generate atto-second XFEL pulses [11]. There is a dispersion bump in the middle of the chicane and two families of sextupole magnets (SF and SD) are located for correcting chromatic aberration in both horizontal and vertical directions. The area enclosed by the dashed line has a mirror-symmetric configuration of magnets and β_0 is also mirror-symmetric.

The results of the correction are summarized in Table 1. The strength of SF and SD is defined by $\frac{B''L}{[B\rho]}$ [m^{-2}], where $B''L$ is the integrated field strength and $[B\rho]$ is the magnetic rigidity. In each column, the upper (lower) value represents the quantity in the horizontal (vertical) direction. Three correction schemes described above are compared, and we see that in any of the schemes (A), (B) or (C), the chromatic aberration is well suppressed and the energy-dependent component of the transfer matrix M_1 can be made small with two families of sextupole magnets. We also see that, in the scheme (C), the matrix M_1 is corrected more effectively than (A) and (B) in the horizontal direction.

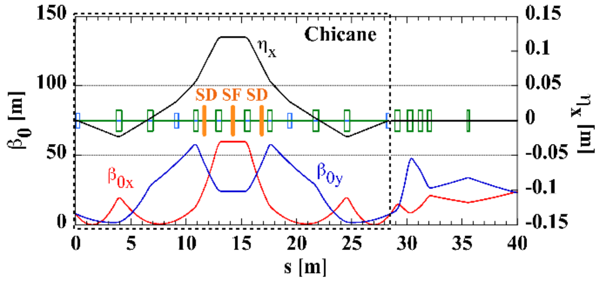

Figure 2: An example lattice with a mirror-symmetric chicane. Blue and green boxes represent dipole and quadrupole magnets, respectively.

Table 1: Comparison of Correction Schemes

	Before Cor.	After Cor.		
		(A) $\xi_{\text{CLOSED}} = 0$	(B) $\xi_{\text{OPEN}} = 0$	(C) $\widetilde{W} = 0$
SF SD	0 .00 0.00	+11.41 -7.74	+11.53 -7.88	+9.71 -7.25
ξ_{CLOSED}	-3.94 -1.73	0.00 0.00	+0.023 +0.052	-0.81 +0.12
ξ_{OPEN}	-3.85 -3.02	-0.023 -0.089	0.00 0.00	-0.81 +0.12
$\widetilde{W}$	39.53 22.30	10.15 1.53	10.44 0.89	0.00 0.00
M_1	$\begin{pmatrix} +74.95 & +153.82 \\ +35.08 & +74.95 \end{pmatrix}$ $\begin{pmatrix} +27.99 & +155.24 \\ +5.09 & +27.99 \end{pmatrix}$	$\begin{pmatrix} -26.52 & -40.92 \\ -17.81 & -26.52 \end{pmatrix}$ $\begin{pmatrix} +1.42 & +6.22 \\ +0.42 & +1.42 \end{pmatrix}$	$\begin{pmatrix} -27.12 & -42.07 \\ -18.13 & -27.12 \end{pmatrix}$ $\begin{pmatrix} +0.61 & +1.70 \\ +0.27 & +0.61 \end{pmatrix}$	$\begin{pmatrix} -5.80 & -1.14 \\ -7.03 & -5.80 \end{pmatrix}$ $\begin{pmatrix} -0.49 & -4.36 \\ +0.07 & -0.49 \end{pmatrix}$

SUMMARY

We have presented a unified formulation of describing the chromatic aberration of both open and closed beam transport systems up to the first order of the relative momentum deviation. This formulation allows us to handle boundary conditions easily, and we have shown that under the periodic boundary conditions, the energy-dependent component of the betatron phase for the open system reduces to the well-known expression of the linear chromaticity used in the closed system such as storage rings. We have also discussed an efficient correction scheme of the chromatic aberration by applying to the example chicane lattice with a mirror-symmetry configuration. A general treatment in the case with other typical boundary conditions is presented in detail in Reference [4].

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