IPAC26

Topology and Non-Linear Matching: From KAM Tori to Beam Profiles

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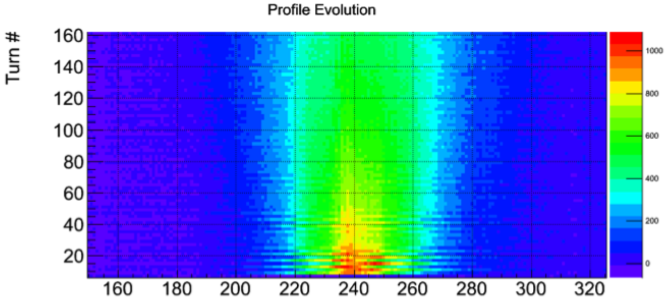
Plan

Beam Filamentation: A Known Problem

- During injection into synchrotrons, **beam-lattice mismatch** can cause beam degradation



Horizontal profile versus turn number for a bunch injected in the LHC [1]

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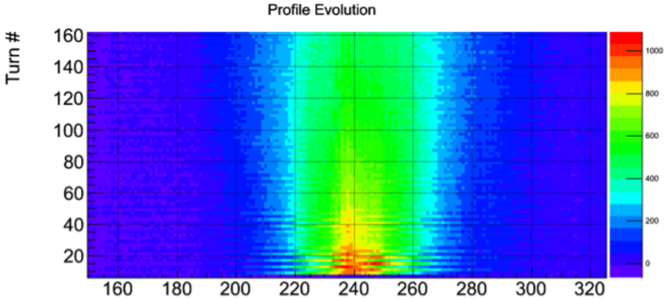

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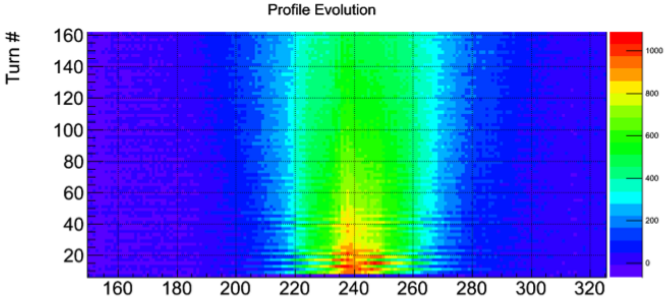

Horizontal profile versus turn number for a bunch injected in the LHC [1]

- In **linear lattices**, problem is well understood (emittance dilution) [2, 3]
- In **non-linear lattices**, emittance alone is not adequate to represent the full picture.

The Challenge: Non-Linear Matching

- **Key questions**: How do an initial beam distributions filament in non-linear lattices? Can we answer this question without tracking the full distribution?
- **Practical concern**: high-brightness colliders (e.g. HL-LHC) are affected by strong lattice non-linearities (high chromaticity, octupoles, beam-beam effects,. . .) and the high amplitude particles are the main responsible for beam losses → **the control of beam transverse tails is paramount** [4, 5, 6]

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  → **find equilibrium distribution** $f_\infty(x, p_x)$
- **Proposed method**: we study this problem using a topological approach, assuming that we are in a phase-space region where the KAM (Kolmogorov–Arnold–Moser) tori persist [7, 8, 9].

Plan

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- KAM theory guarantees quasi-periodic motion exists in specific phase-space regions. It is **naturally used for the long-term stability** and its application is difficult for realistic lattice configuration.

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- We are interested on the beam **filamentation** (single particle effects) not particle **diffusion** (collective effects, noise, ...). The two time scales separate naturally in the system:
  - **filamentation**: $\tau_{\mathcal{F}} \sim$ hundred turns (fast)
  - **diffusion**: $\tau_{\mathcal{D}} \sim$ tens of minutes (slow)
- We will use **KAM theory from a (short-term, few $\tau_{\mathcal{F}}$) topological perspective** and not for the long-term stability problem.

A step-by-step example

We will consider*

1. a simple 2D non-linear lattice (Hénon map in *normalised coordinates* [8])

$$z_{N+1} = e^{i2\pi Q_x}\left[z_N - \frac{i}{4}\left(z_N + z_N^*\right)^2\right], \qquad (1)$$

where $z_N = x_N - i\ p_{x,N}$ is the phase space position at turn N and Q_x is the map's tune †

*A python script is available here based on [10, 11].
†We set $Q_x = 0.2071$.
‡We set r=0.33 $\sqrt{\mathrm{m}}$.

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2. a uniform beam distribution‡

$$f_0 = \begin{cases} \dfrac{1}{\pi r^2}, & \text{if } x^2 + p_x^2 < r^2, \\ 0, & \text{otherwise.} \end{cases} \tag{2}$$

---

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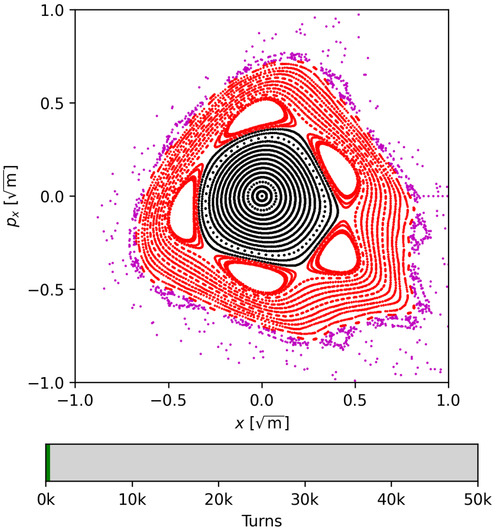
- The stability of the phase space orbit within the time window of interest determines our “KAM region”.



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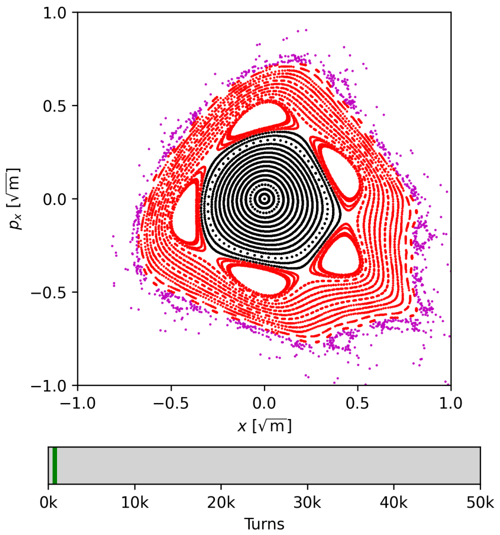

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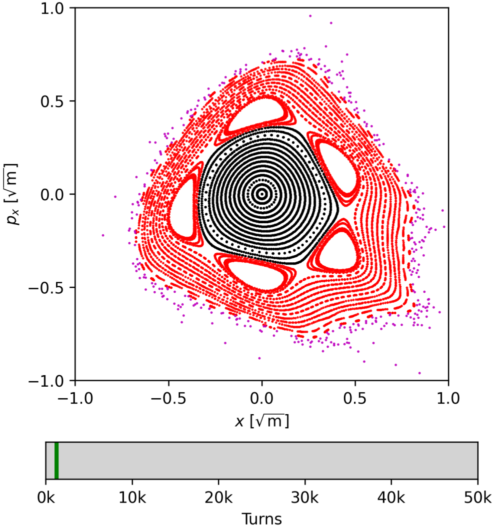

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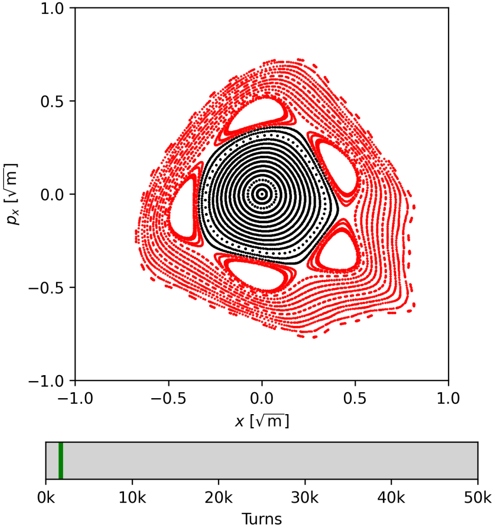

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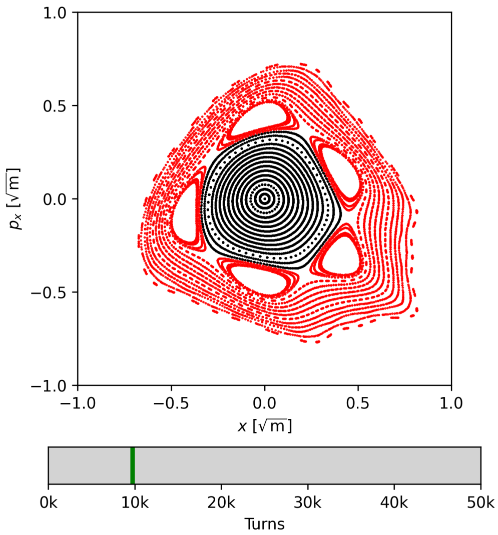

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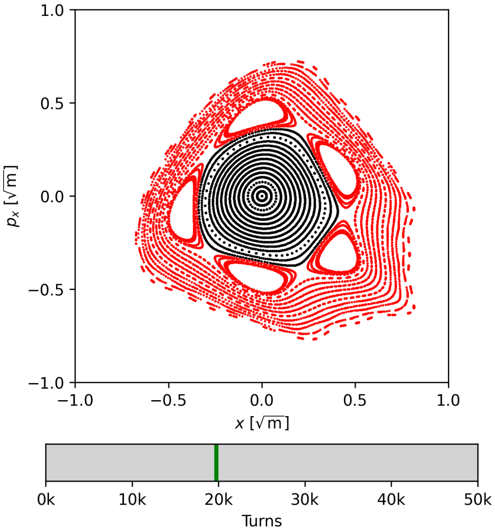

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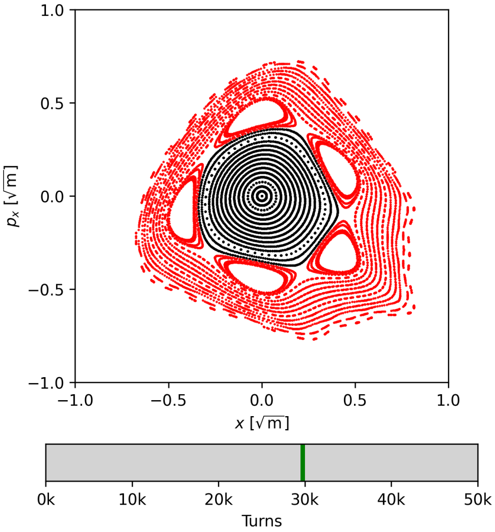

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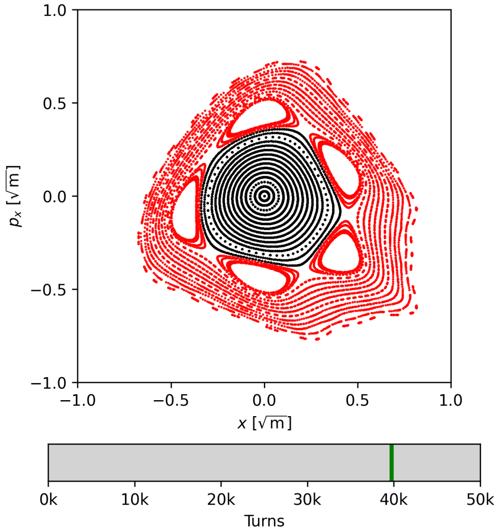

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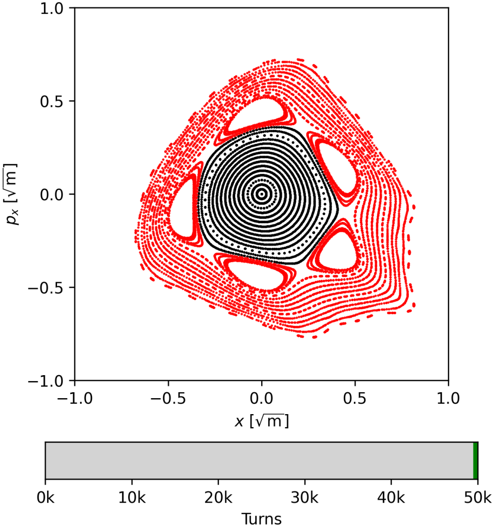

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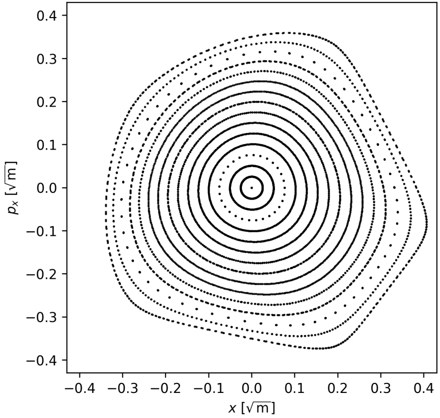

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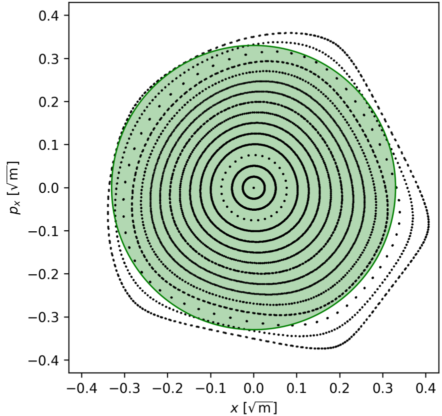
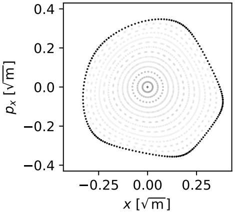
- The initial distribution will be fully enclosed in the non-resonant KAM region.



From Tracking to Torus

1. Track few particles (e.g. 20) for few turns (e.g. 5k)



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From Tracking to Torus

1. Track few particles (e.g. 20) for few turns (e.g. 5k)
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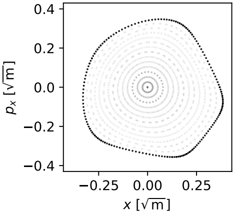

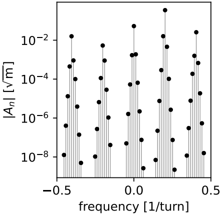

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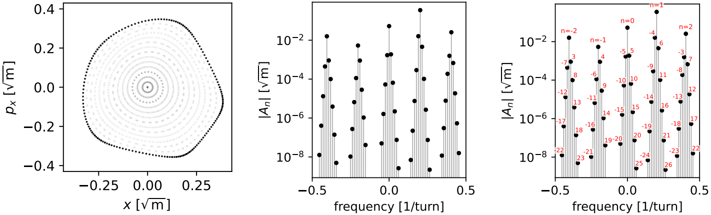

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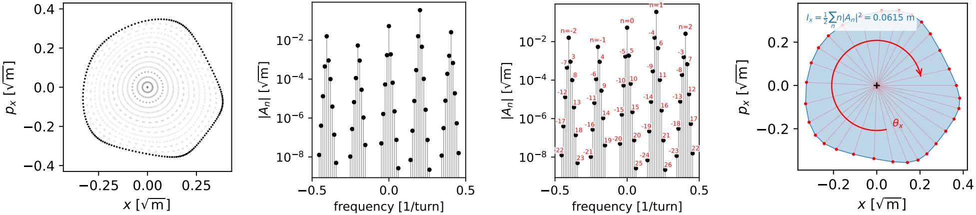
3. Use Frequency Identification, **FI**, to sort the different harmonics [15]
4. → Build an **analytical I_x, θ_x torus representation** [9]:

$$x(N) - ip_x(N) = \sum_n A_n e^{i\,2\pi Q_n N} \overset{\textbf{FI}}{=} \sum_n A_n e^{i\,n\overbrace{2\pi Q_x N}^{}_{\theta_x}}, \qquad (3)$$



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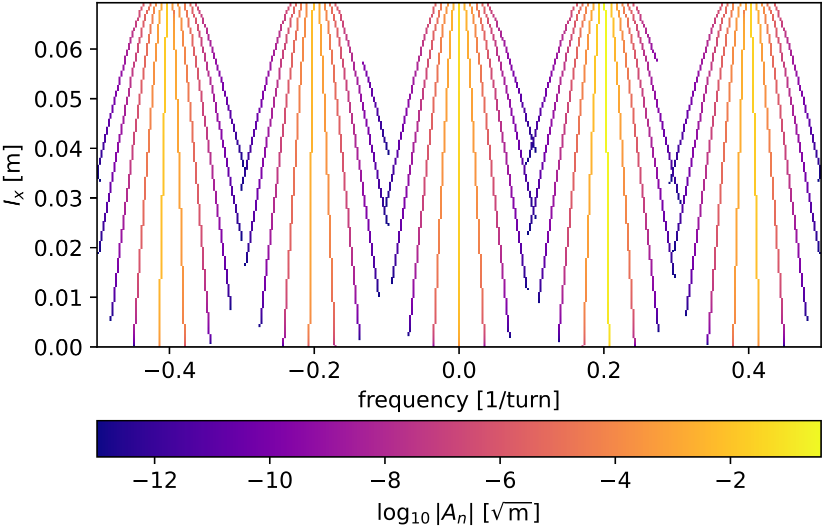
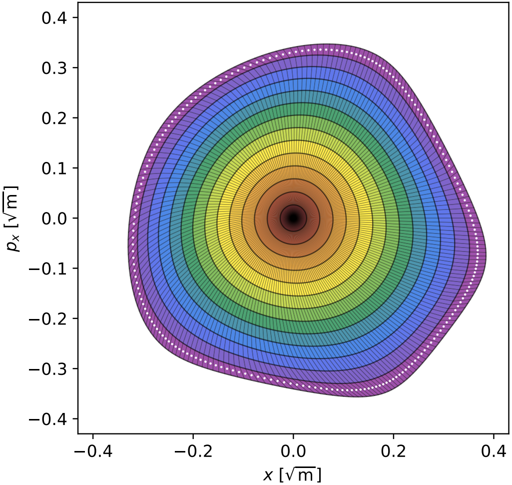
Topology Spectral Content and Amplitude Detuning



From Torus to Tori Foliation

- Each particle maps to a **KAM Torus**
- A **Tori Foliation meshes**, via a $\{I_x, \theta_x\}$ mapping, the region of the phase space we are interested in.



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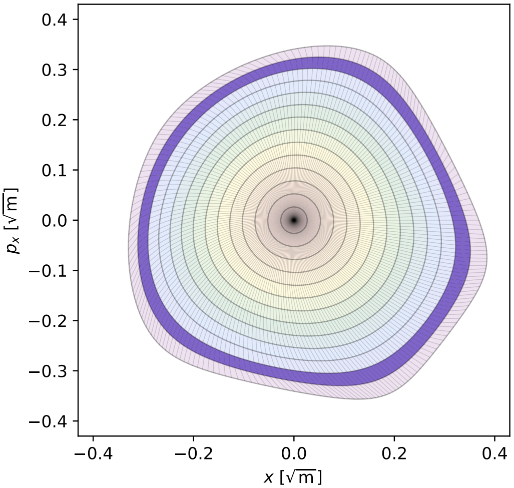

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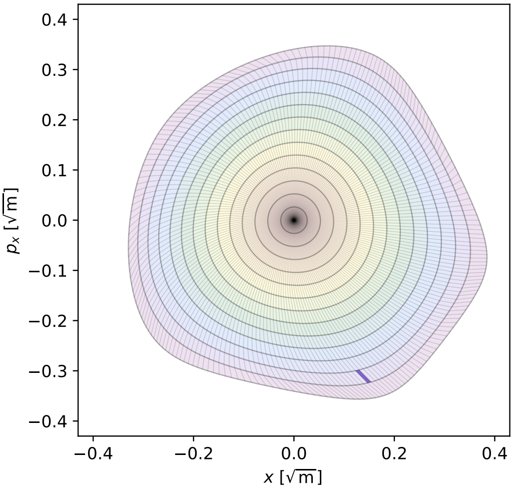
- Slabs are divided into **cells** with area $d\mathcal{A} = dI_x d\theta_x$
  - $\longrightarrow$ cells belonging to the same slab have (at the limit) same $d\mathcal{A}$



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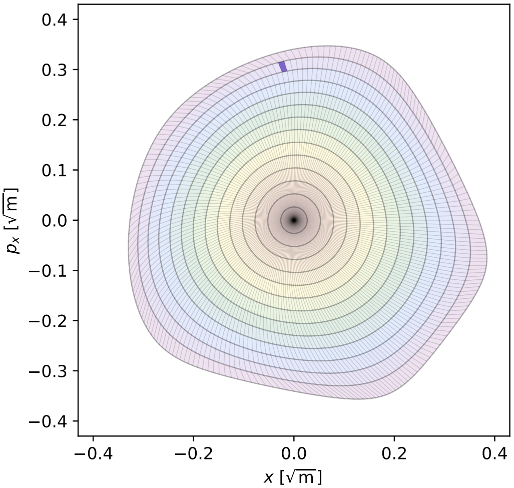

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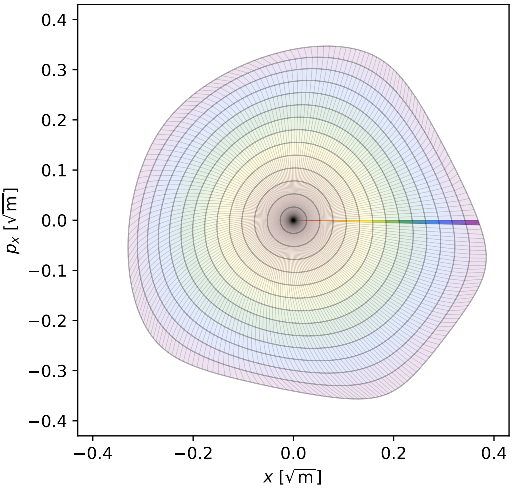
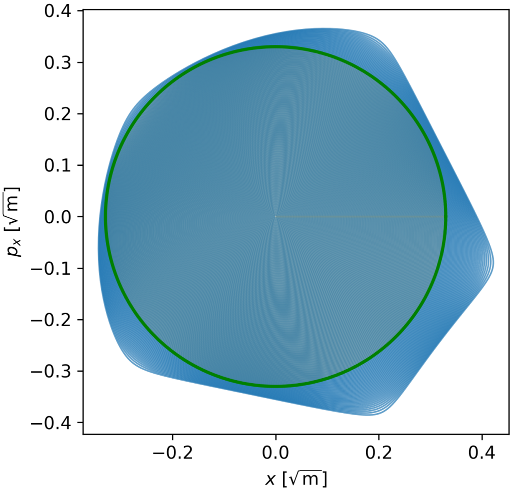
  - $\longrightarrow$ cells belonging to different slabs have (in general) different $d\mathcal{A}$



Plan

Initial Distribution on the Mesh

- We represent the beam as cell-centered density on the mesh provided by the tori foliation



Initial Distribution on the Mesh

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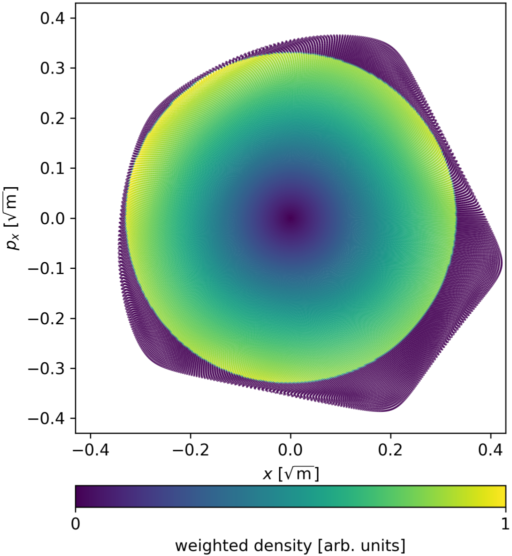
- Due to the fact that area is cell-dependent, we need to **weight the by density cell area $\mathcal{A}_c$**:

$$f_{0,c} \propto \mathcal{A}_c \sum_{v \in \mathcal{V}(c)} f_0(v)$$

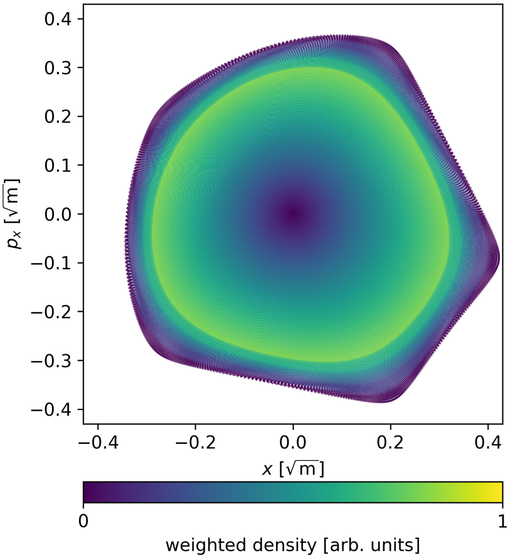
where $\mathcal{V}(c)$ are the cell vertices.



The Filamentation Process

Filamentation occurs by **amplitude detuning**:

- Within the same slab, the θ_x-dependence washes out.
- After filamentation, the weighed density becomes θ_x-invariant (slab-averaged)



Matched Profiles: Tails Develop

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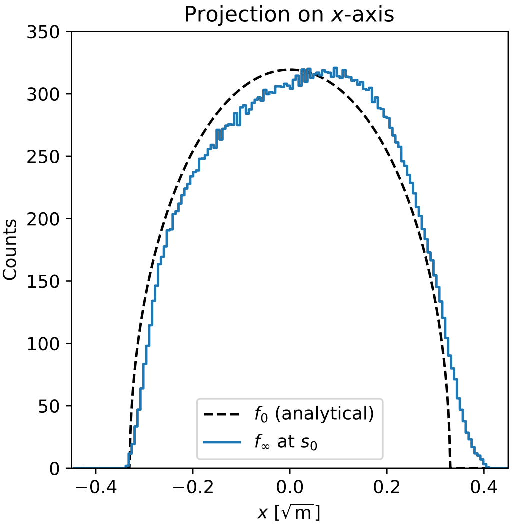

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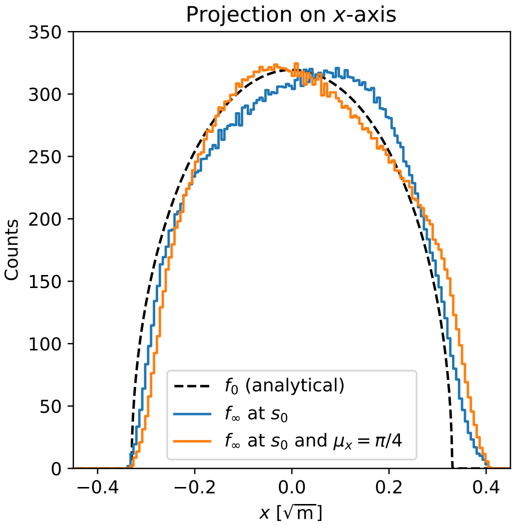

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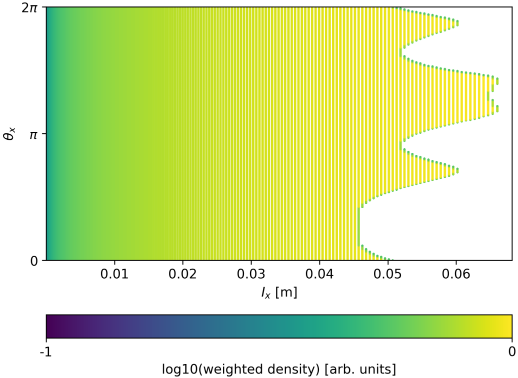
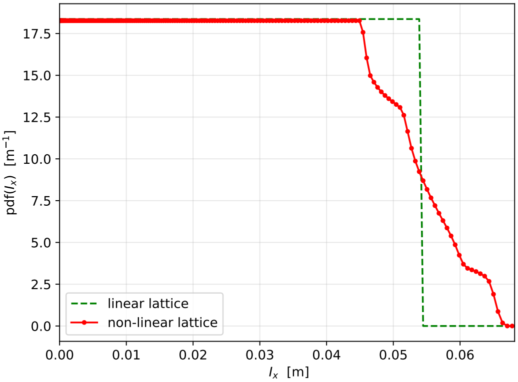


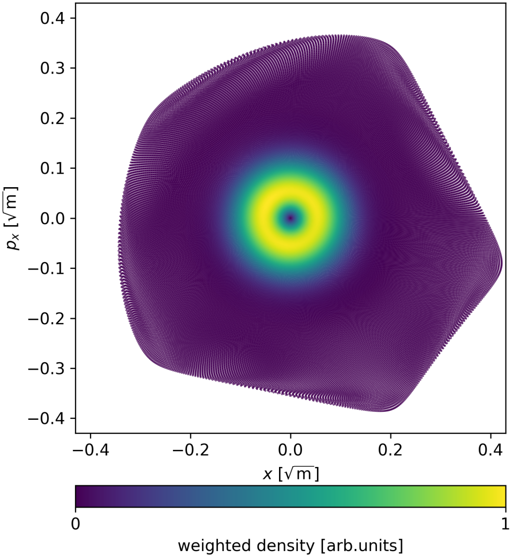
Matched Profiles: Tails Develop

- Profiles develop **asymmetric tails**
- Profiles are NOT s-invariant
- The action distribution pdf(I_x) is **conserved** (symplectic invariant)



Gaussian Beam Case

- With a very similar approach we can consider other initial distribution, e.g. a truncated Gaussian ($\sigma = 0.055$, $6\ \sigma$ truncation)



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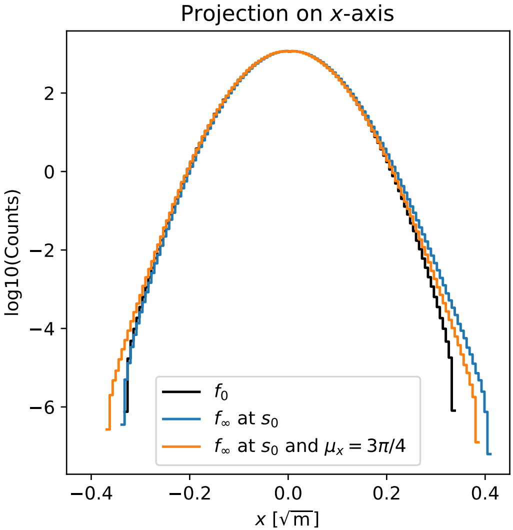

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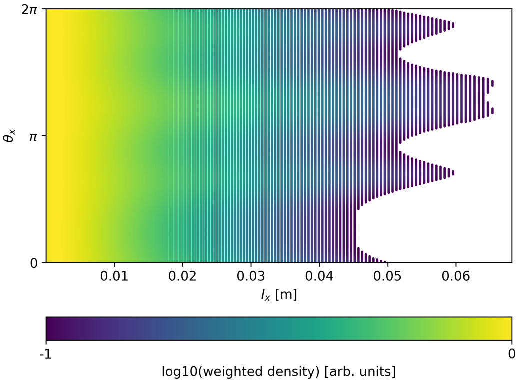

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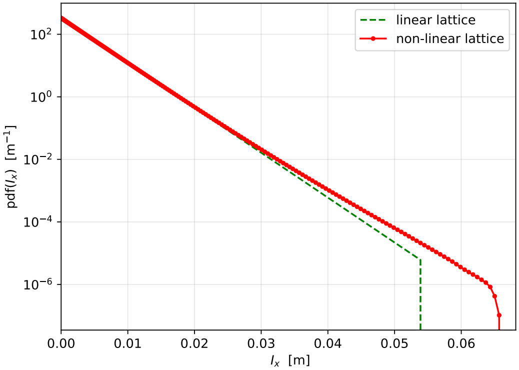
- **Heavier tails** develop through filamentation
- Effect is visible for intermediate amplitudes



Plan

Conclusions and Outlook

- **KAM Tori Foliation** provides a natural framework for meshing the phase space and investigating problems complementary to those related to the long-term stability of the orbit of particles.
- We discussed an application: **quantify how beams develop tails** through filamentation
- Transverse profiles **change along lattice**: only the action distribution, pdf(I_x), is preserved along the non-linear lattice and should be used as a metric of the beam distribution
- In non-linear dynamics, the tori foliation together with the pdf(I_x) can provide a **natural factorization between lattice and beam properties**, respectively.
- This formalism can be naturally extended to 4D and 6D motion and can be used to generalise the Abel transform to non-linear lattice.

Thank you for your attention.

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Backup slides

Generalizing the Abel Transform

- The Classical Abel Transform works for 2D linear systems. It allows reconstructing the matched 2D distribution given its x-profile. Under given assumptions, can be generalised to 4D and 6D [16]
- In the linear case, slab projections follow $\propto \frac{1}{\sqrt{r^2 - x^2}}$ (the kernel of the Abel transform)
- We can build a set of slab-projections for the non-linear case and inverse numerically the problem: reconstruct the distribution from the measured x-profiles, e.g. using non-negative least-squares.