



Approximate Invariant Analysis: An Efficient Framework for Nonlinear Beam Dynamic



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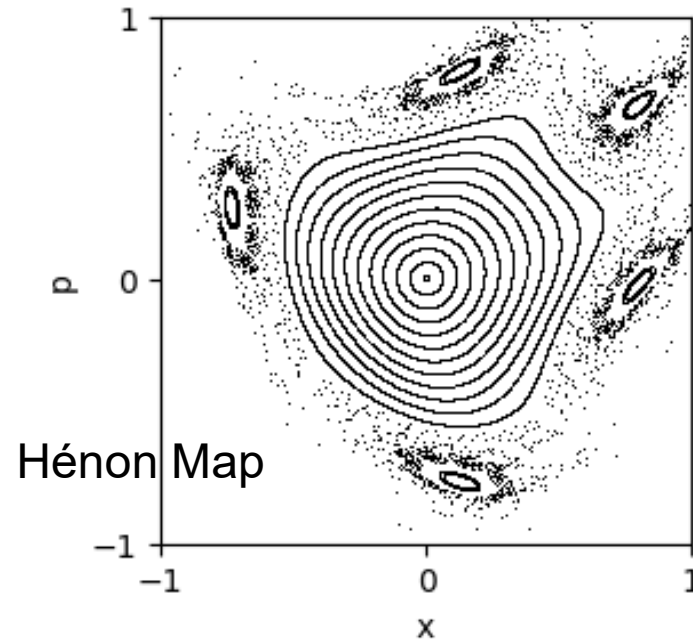
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17th International Particle Accelerator Conference, Deauville, Normandy, France

Outline

- Motivation
- Construction of Approximate Invariants (AI)
- Applications on Hénon map and NSLS-II ring
- Extraction of Betatron Tune
- Summary

Motivation



For **nonintegrable** Hamiltonian systems, **KAM** theorem proves **stable quasi-periodic** motions persist under small perturbations, **Approximate Invariants (AI)** form distorted but unbroken **tori**.

Constructing invariants

1. Build one-turn-map using Truncated Power Series Algorithm (TPSA)
2. Re-write map as square matrix (SM)
3. Implement eigen-analysis on transposed SM **order-by-order** by utilizing its **special sparsity**

$$x = m_{1,1000}x + m_{1,0100}p_x + m_{1,0010}y + m_{1,0001}p_y + \cdots + m_{1,000\Omega}p_y^\Omega$$

$$Z = [x, p_x, y, p_y, x^2, xp_x, xy, xp_y, p_x^2, p_x p_y, \cdots, p_y^\Omega]^T$$

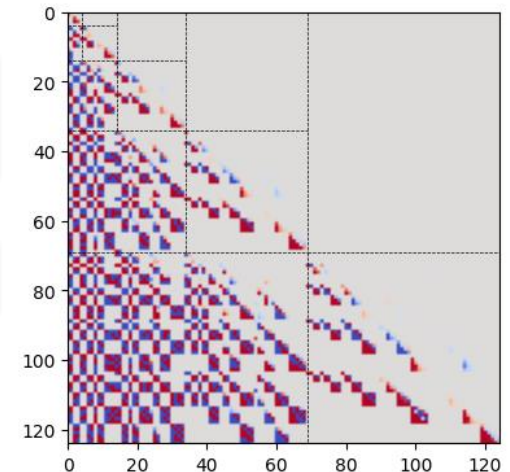
$$Z_{final} = M Z_{initial}$$

$$\mathcal{K}^{(\Omega)} = V^T Z = \sum_{abcd} c_{abcd} x^a p_x^b y^c p_y^d$$

$$V^T Z = V^T M Z \implies V = M^T V$$

$$\begin{bmatrix} V^{(i)} \\ V^{(i+1)} \end{bmatrix} = \begin{bmatrix} m_{11} & \mathbf{0} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} V^{(i)} \\ V^{(i+1)} \end{bmatrix}$$

$$V^{(i+1)} = (I - m_{22})^{-1} m_{21} V^{(i)}$$



Y. Li, D. Xu and Y. Hao, Phys. Rev. Accel. Beams 28, 074001

DOI: <https://doi.org/10.1103/m349-wmnr>

Note that the methods of AI construction are not unique!

Example 1: 1D Hénon Map

$$\begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} \cos(2\pi\nu) & \sin(2\pi\nu) \\ -\sin(2\pi\nu) & \cos(2\pi\nu) \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{k_2 l}{2} x^2 \end{bmatrix}$$

I	V [993]	Base [21 / 21]
1	1.0000000000000000e+00	2 0
2	-9.8065510000000001e-18	1 1
3	1.0000000000000000e+00	0 2
4	-1.6475940000000000e-01	3 0
5	-1.0000000000000000e+00	2 1
6	-7.1451410000000000e-01	1 2
7	7.2713580000000000e-18	0 3
8	-1.5363720000000000e-01	4 0
9	7.2651140000000000e-01	3 1
10	1.4414280000000000e+00	2 2
11	2.6432850000000000e-16	1 3
12	-1.9122030000000000e-01	0 4
13	-3.8140080000000000e-01	5 0
14	-1.4342180000000000e+00	4 1
15	2.9174010000000000e+00	3 2
16	3.7790470000000000e-01	2 3
17	-2.3204860000000000e+00	1 4
18	6.7164000000000000e-03	0 5

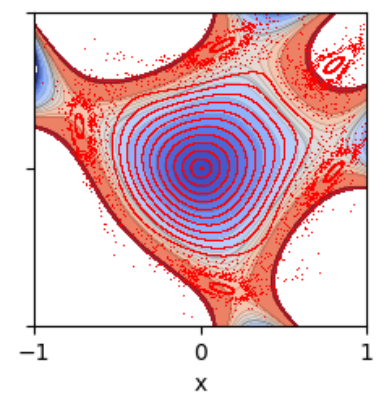
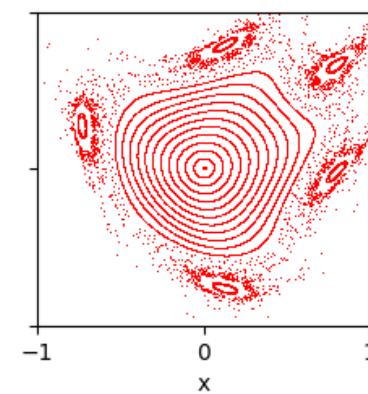
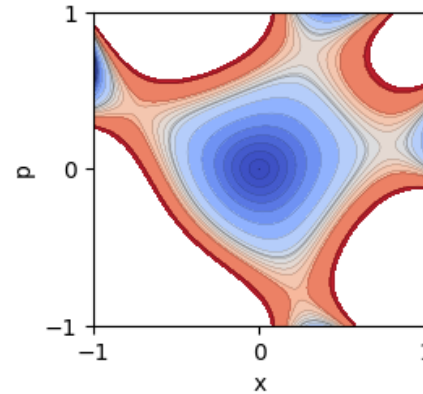
x^2
 \vdots
 $x^2 p^2$
 \vdots

$\nu = 0.22$

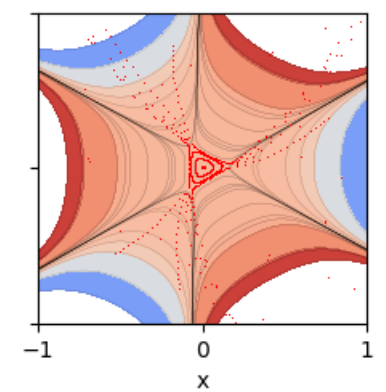
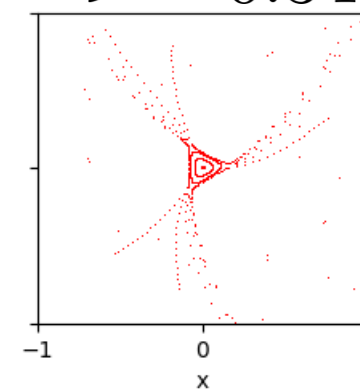
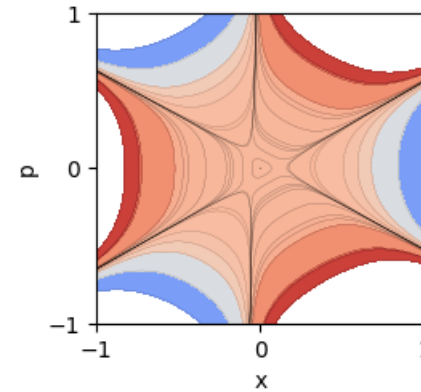
AI contour

simulation

overlap

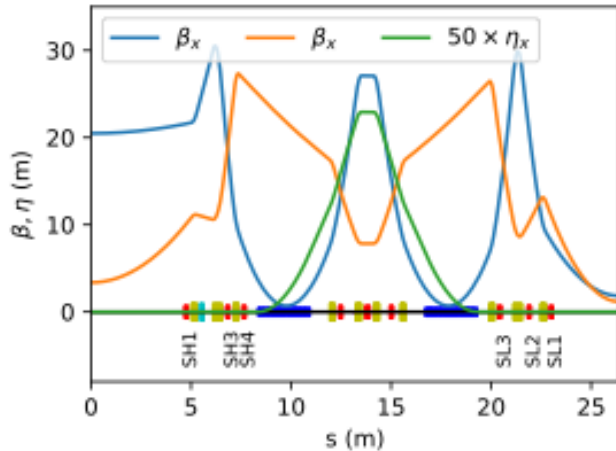


$\nu = 0.34$



Example 2: NSLS-II

2DoF system: Two AIs



I	V [447]	Base
1	2.3852920000000000e-03	2 0 0 0
2	3.1857910000000000e-18	1 1 0 0
3	9.9999720000000000e-01	0 2 0 0
4	6.4826930000000000e-03	3 0 0 0
5	-2.9686490000000000e-02	2 1 0 0
6	-4.6222720000000000e+00	1 2 0 0
7	8.4559730000000000e-02	1 0 2 0
8	3.9653700000000000e-02	1 0 1 1
9	1.7754430000000000e+00	1 0 0 2
10	7.3242720000000000e+00	0 3 0 0
11	-7.5897390000000000e-02	0 1 2 0
12	1.5835300000000000e+01	0 1 1 1
13	2.1344770000000000e+00	0 1 0 2
14	1.1194590000000000e-01	4 0 0 0
15	-2.6589190000000000e-01	3 1 0 0
16	-6.3821710000000000e+02	2 2 0 0
17	5.1735570000000000e+00	2 0 2 0
18	3.1148610000000000e-01	2 0 1 1
19	3.1859250000000000e+01	2 0 0 2
20	-1.3876010000000000e+01	1 3 0 0
21	1.3903490000000000e+00	1 1 2 0
22	6.2439340000000000e+02	1 1 1 1
23	-1.2438570000000000e+01	1 1 0 2
24	4.3607430000000000e+03	0 4 0 0
25	7.3083500000000000e+02	0 2 2 0
26	-1.8898160000000000e+02	0 2 1 1
27	3.8326430000000000e+04	0 2 0 2
28	3.7500000000000000e+00	0 0 4 0
29	-1.5063120000000000e-01	0 0 3 1
30	1.6200000000000000e+02	0 0 2 2

I	V [4]	Base
1	-0.000000e+00	1 0 0 0
2	-0.000000e+00	0 1 0 0
3	-0.000000e+00	0 0 1 0
4	-0.000000e+00	0 0 0 1
5	-0.000000e+00	2 0 0 0
6	-0.000000e+00	1 1 0 0
7	-0.000000e+00	1 0 1 0
8	-0.000000e+00	1 0 0 1
9	-0.000000e+00	0 2 0 0
10	-0.000000e+00	0 1 1 0
11	-0.000000e+00	0 1 0 1
12	2.969947e-01	0 0 2 0
13	1.121019e-14	0 0 1 1
14	3.367063e+00	0 0 0 2
15	0.000000e+00	3 0 0 0
16	0.000000e+00	2 1 0 0
17	0.000000e+00	2 0 1 0
18	0.000000e+00	2 0 0 1
19	0.000000e+00	1 2 0 0
20	0.000000e+00	1 1 1 0
21	0.000000e+00	1 1 0 1
22	-4.703013e+00	1 0 2 0
23	-1.778965e+00	1 0 1 1
24	5.331860e+01	1 0 0 2
25	0.000000e+00	0 3 0 0
26	0.000000e+00	0 2 1 0
27	0.000000e+00	0 2 0 1
28	4.937315e+00	0 1 2 0
29	2.033959e+02	0 1 1 1
30	-5.597491e+01	0 1 0 2

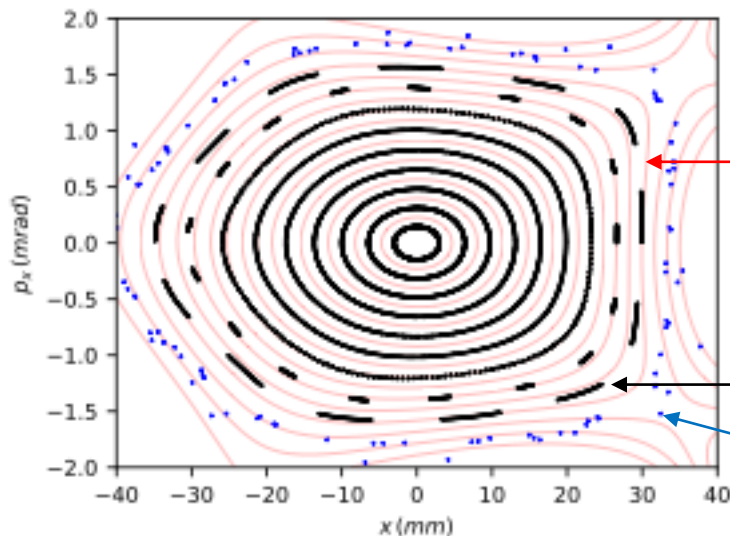
Scaled linear
Courant-Snyder
invariants

All information - Tori profile, frequency and stability are hidden in AIs!

Phase Space at Mid-Plane

- Can't visualize in 4D space, so let's project to 2D mid-plane

$$\mathcal{K} = \sum_{ab} c_{ab} x^a p_x^b$$



Red lines: approximate invariant contours
If closed (torus), motion is stable
Outermost torus is **dynamic aperture boundary**

Black dots: simulated stable motion

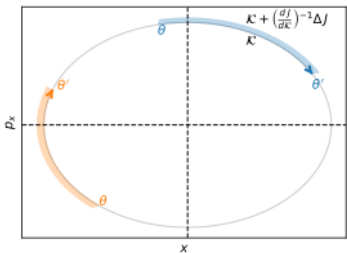
Blue dots: simulated unstable motion

Mid-plane with $y = p_y = 0$

Betatron Tune's Fractional Part

Poincaré Rotation Number (PRN)

Poincaré map



$$\nu_i = \frac{\int_{x'}^x \left(\frac{\partial \mathcal{K}}{\partial p_x}\right)^{-1} dx}{\oint \left(\frac{\partial \mathcal{K}}{\partial p_x}\right)^{-1} dx} = \frac{dJ'}{d\mathcal{K}} \bigg/ \frac{dJ}{d\mathcal{K}} \approx \frac{\Delta J'}{\Delta \mathcal{K}} \bigg/ \frac{\Delta J}{\Delta \mathcal{K}}$$

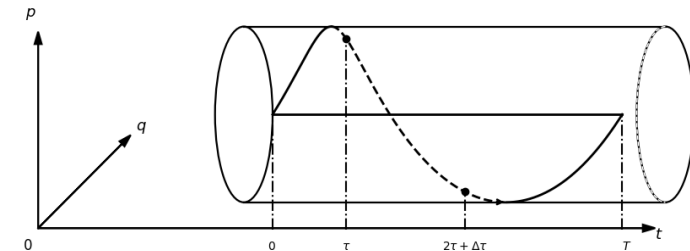
$S = 2\pi J$ and J : action

S. Nagaitsev and T. Zolkin, Phys. Rev. Accel. Beams 23, 054001
 C. Mitchell, R. Ryne, K. Hwang, S. Nagaitsev and T. Zolkin, Phys. Rev. E 103, 062216

Quasiperiodic tune

$$\nu = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{i=1}^N \nu_i$$

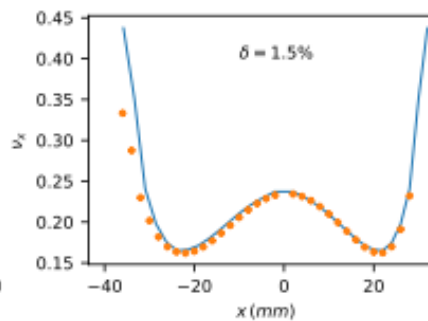
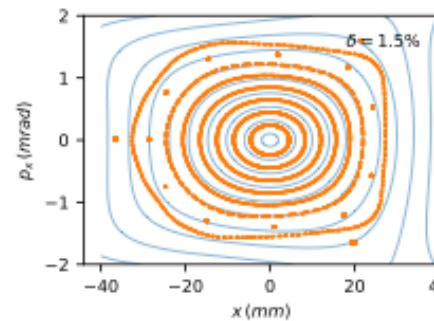
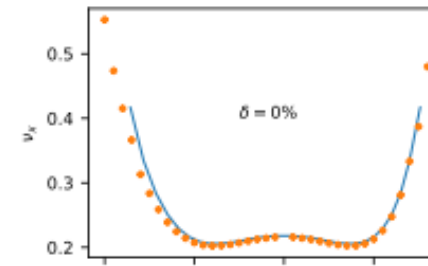
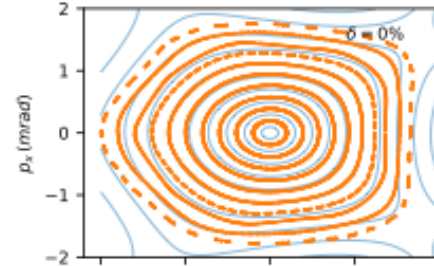
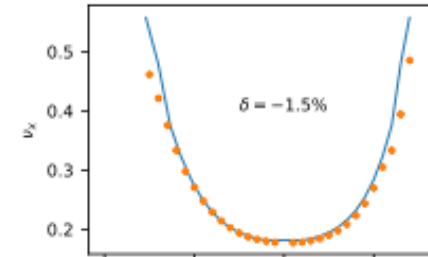
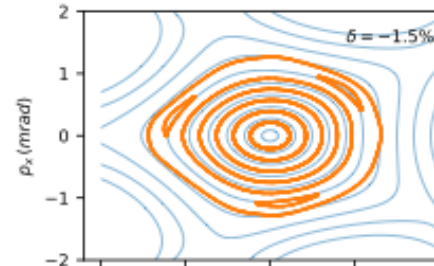
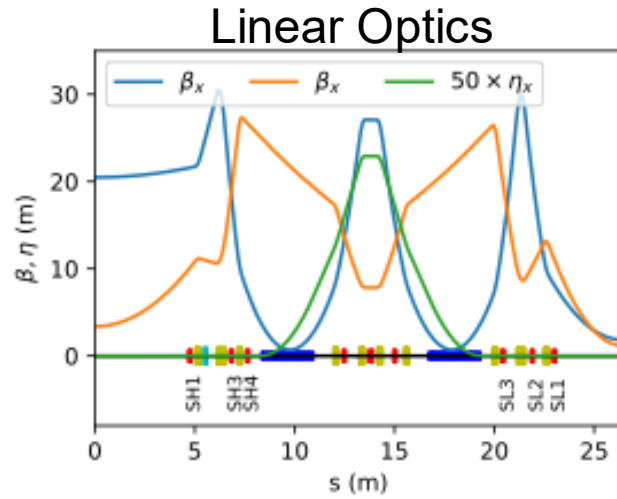
Time-of-Flight of effective Hamiltonian



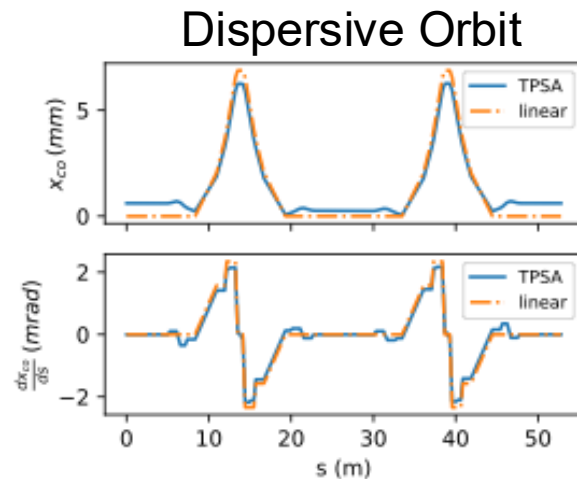
$$\nu_i = \frac{\tau}{T}$$

D. Xu, Y. Li, Y. Hao and S. Nagaitsev
 arXiv:2512.16060

Amplitude-Dependent Detuning (ADD)



Both DAs and ADDs reasonably agree with tracking simulation

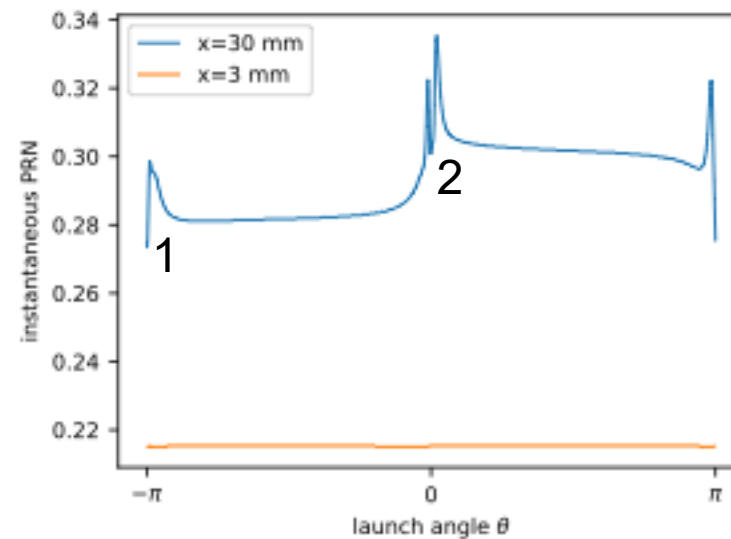
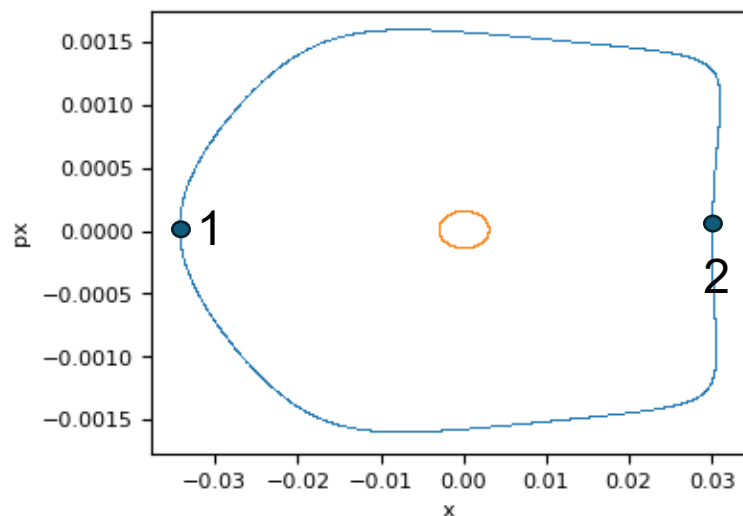


Blue lines: AIA

Yellow dots: tracking simulation

Tune diffusion

Tune diffusion: **Sensitivity** on initial condition

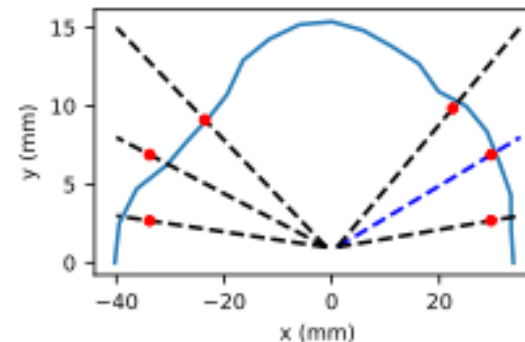
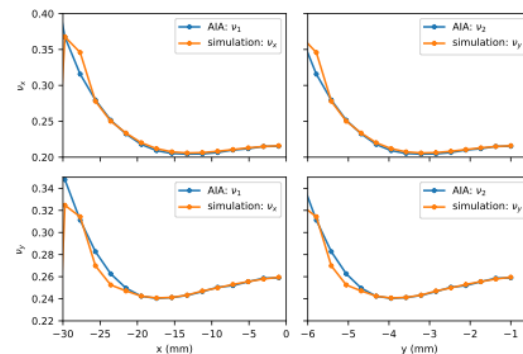


Once quasi-periodicity is lost, the motion become unstable.

Summary

- A new nonlinear beam dynamics framework
- Most existing mathematical tools, such as Hamiltonian perturbation, canonical transformation, Lie algebra technique, normal form, even Courant-Snyder parameterization etc. are **NOT** needed
- Dynamic aperture and fractional tune can be estimated quite accurately
- Although only 1DoF is demonstrated here, we already extend it to 2DoF

Tune-shift-with-amplitude



Dynamic aperture

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Questions?



LOGO