# Approximate Invariant Analysis:
## An Efficient Framework for Nonlinear Beam Dynamic

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# Outline

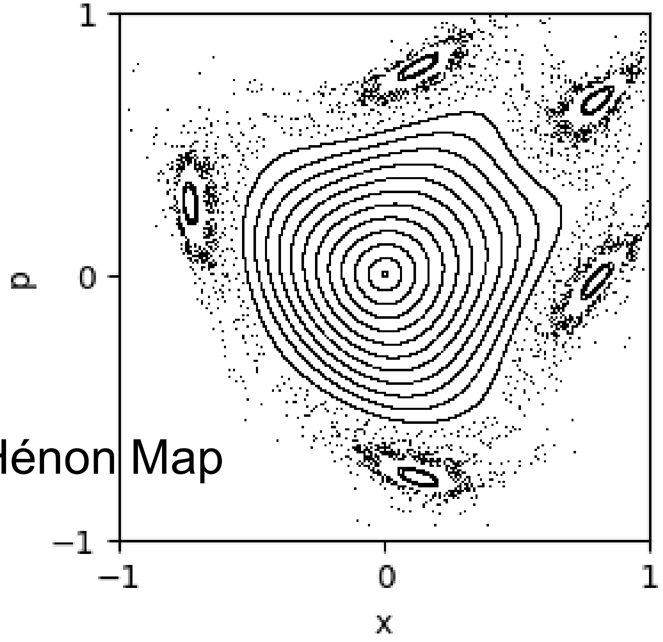
- Motivation
- Construction of Approximate Invariants (AI)
- Applications on Hénon map and NSLS-II ring
- Extraction of Betatron Tune
- Summary

# Motivation

How to answer the question “Can you explain how to optimize dynamic aperture (DA)?”

Current nonlinear beam dynamics framework is too complicated!

# Motivation




Hénon Map

For **nonintegrable** Hamiltonian systems, **KAM** theorem proves **stable quasi-periodic** motions persist under small perturbations, **Approximate Invariants (AI)** form distorted but unbroken **tori**.

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# Constructing invariants

1. Build one-turn-map using Truncated Power Series Algorithm (TPSA)
2. Re-write map as square matrix (SM)
3. Implement eigen-analysis on transposed SM **order-by-order** by utilizing its **special sparsity**

$$x = m_{1,1000}x + m_{1,0100}p_x + m_{1,0010}y + m_{1,0001}p_y + \cdots + m_{1,000\Omega}p_y^{\Omega}$$

$$Z = [x, p_x, y, p_y, x^2, xp_x, xy, xp_y, p_x^2, p_xy, \cdots, p_y^{\Omega}]^T$$

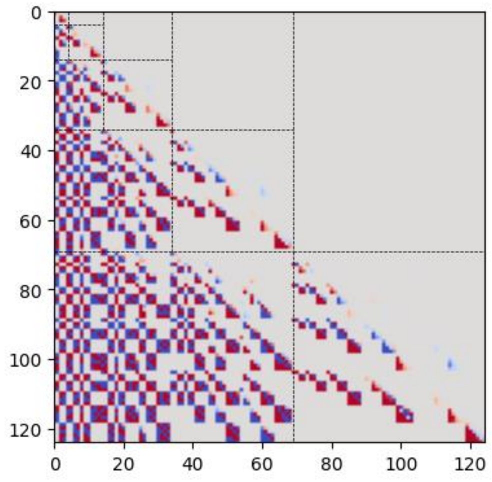
$$Z_{final} = MZ_{initial}$$

$$\mathcal{K}^{(\Omega)} = V^T Z = \sum_{abcd} c_{abcd}x^a p_x^b y^c p_y^d$$

$$V^T Z = V^T M Z \Longrightarrow V = M^T V$$

$$\begin{bmatrix} V^{(i)} \\ V^{(i+1)} \end{bmatrix} = \begin{bmatrix} m_{11} & \mathbf{0} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} V^{(i)} \\ V^{(i+1)} \end{bmatrix}$$

$$V^{(i+1)} = (I - m_{22})^{-1} m_{21} V^{(i)}$$




**Y. Li, D. Xu and Y. Hao, Phys. Rev. Accel. Beams 28, 074001**
**DOI: https://doi.org/10.1103/m349-wmnr**

Note that the methods of AI construction are not unique!

# Example 1: 1D Hénon Map

$$\begin{bmatrix} x \\ p \end{bmatrix} = \begin{bmatrix} \cos(2\pi\nu) & \sin(2\pi\nu) \\ -\sin(2\pi\nu) & \cos(2\pi\nu) \end{bmatrix} \begin{bmatrix} x \\ p \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{k_2 l}{2} x^2 \end{bmatrix}$$

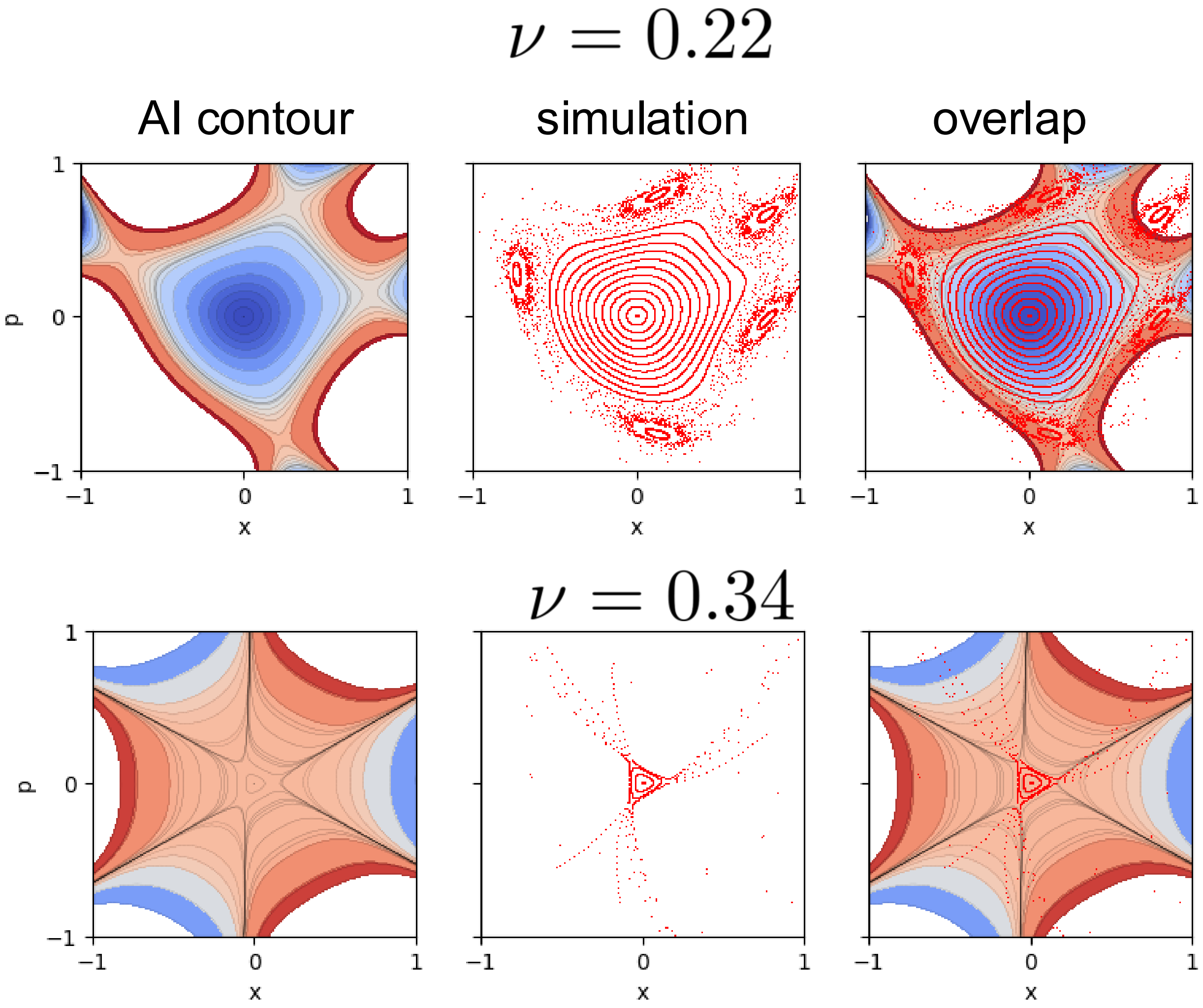
| I | V [993] | Base | | [ 21 / 21 ] |
|---|---|---|---|---|
| 1 | 1.000000000000000e+00 | 2 | 0 | 3 |
| 2 | -9.806551000000001e-18 | 1 | 1 | 4 |
| 3 | 1.000000000000000e+00 | 0 | 2 | 5 |
| 4 | -1.647594000000000e-01 | 3 | 0 | 6 |
| 5 | -1.000000000000000e+00 | 2 | 1 | 7 |
| 6 | -7.145141000000000e-01 | 1 | 2 | 8 |
| 7 | 7.271358000000000e-18 | 0 | 3 | 9 |
| 8 | -1.536372000000000e-01 | 4 | 0 | 10 |
| 9 | 7.265114000000000e-01 | 3 | 1 | 11 |
| 10 | 1.441428000000000e+00 | 2 | 2 | 12 |
| 11 | 2.643285000000000e-16 | 1 | 3 | 13 |
| 12 | -1.912203000000000e-01 | 0 | 4 | 14 |
| 13 | -3.814008000000000e-01 | 5 | 0 | 15 |
| 14 | -1.434218000000000e+00 | 4 | 1 | 16 |
| 15 | 2.917401000000000e+00 | 3 | 2 | 17 |
| 16 | 3.779047000000000e-01 | 2 | 3 | 18 |
| 17 | -2.320486000000000e+00 | 1 | 4 | 19 |
| 18 | 6.716400000000000e-03 | 0 | 5 | 20 |

$x^2$ (row 1)

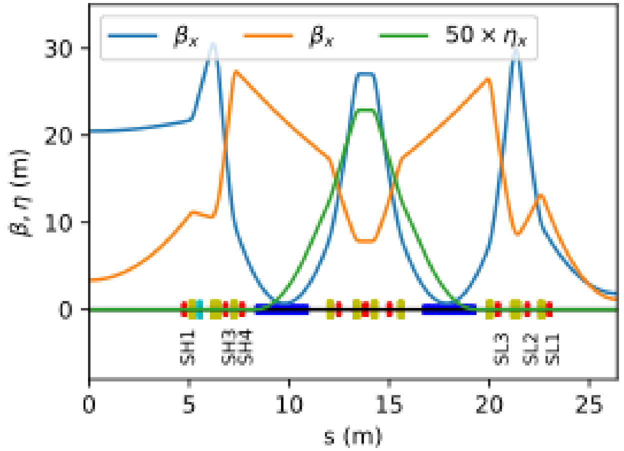
⋮

$x^2p^2$ (row 10)

⋮



# Example 2: NSLS-II




2DoF system: Two AIs

| I | V [447] | Base |
|---|---|---|
| 1 | 2.385292000000000e-03 | 2 0 0 0 |
| 2 | 3.185791000000000e-18 | 1 1 0 0 |
| 3 | 9.999972000000000e-01 | 0 2 0 0 |
| 4 | 6.482693000000000e-03 | 3 0 0 0 |
| 5 | -2.968649000000000e-02 | 2 1 0 0 |
| 6 | -4.622272000000000e+00 | 1 2 0 0 |
| 7 | 8.455973000000000e-02 | 1 0 2 0 |
| 8 | 3.965370000000000e-02 | 1 0 1 1 |
| 9 | 1.775443000000000e+00 | 1 0 0 2 |
| 10 | 7.324272000000000e+00 | 0 3 0 0 |
| 11 | -7.589739000000000e-02 | 0 1 2 0 |
| 12 | 1.583530000000000e+01 | 0 1 1 1 |
| 13 | 2.134477000000000e+00 | 0 1 0 2 |
| 14 | 1.119459000000000e-01 | 4 0 0 0 |
| 15 | -2.658919000000000e-01 | 3 1 0 0 |
| 16 | -6.382171000000000e+02 | 2 2 0 0 |
| 17 | 5.173557000000000e+00 | 2 0 2 0 |
| 18 | 3.114861000000000e-01 | 2 0 1 1 |
| 19 | 3.185925000000000e+01 | 2 0 0 2 |
| 20 | -1.387601000000000e+01 | 1 3 0 0 |
| 21 | 1.390349000000000e+00 | 1 1 2 0 |
| 22 | 6.243934000000000e+02 | 1 1 1 1 |
| 23 | -1.243857000000000e+01 | 1 1 0 2 |
| 24 | 4.360743000000000e+03 | 0 4 0 0 |
| 25 | 7.308350000000000e+02 | 0 2 2 0 |
| 26 | -1.889816000000000e+02 | 0 2 1 1 |
| 27 | 3.832643000000000e+04 | 0 2 0 2 |
| 28 | 3.750000000000000e+00 | 0 0 4 0 |
| 29 | -1.506312000000000e-01 | 0 0 3 1 |
| 30 | 1.620000000000000e+02 | 0 0 2 2 |

Scaled linear Courant-Synder invariants

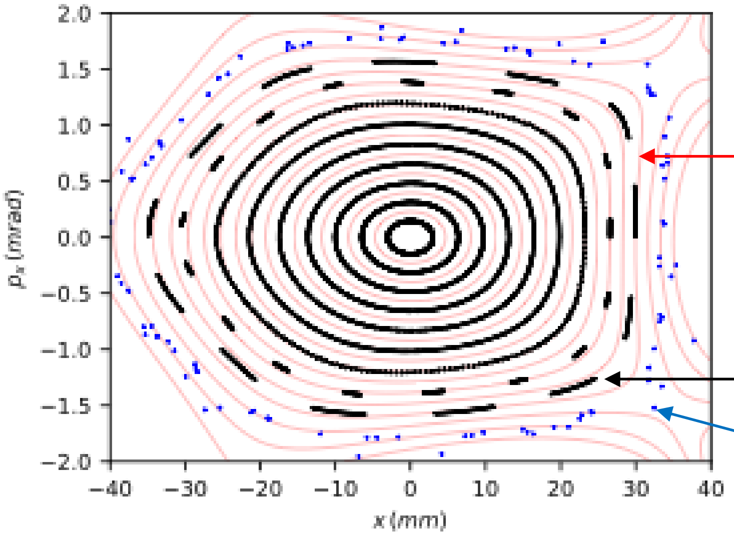
| I | V[4] | Base |
|---|---|---|
| 1 | -0.000000e+00 | 1 0 0 0 |
| 2 | -0.000000e+00 | 0 1 0 0 |
| 3 | -0.000000e+00 | 0 0 1 0 |
| 4 | -0.000000e+00 | 0 0 0 1 |
| 5 | -0.000000e+00 | 2 0 0 0 |
| 6 | -0.000000e+00 | 1 1 0 0 |
| 7 | -0.000000e+00 | 1 0 1 0 |
| 8 | -0.000000e+00 | 1 0 0 1 |
| 9 | -0.000000e+00 | 0 2 0 0 |
| 10 | -0.000000e+00 | 0 1 1 0 |
| 11 | -0.000000e+00 | 0 1 0 1 |
| 12 | 2.969947e-01 | 0 0 2 0 |
| 13 | 1.121019e-14 | 0 0 1 1 |
| 14 | 3.367063e+00 | 0 0 0 2 |
| 15 | 0.000000e+00 | 3 0 0 0 |
| 16 | 0.000000e+00 | 2 1 0 0 |
| 17 | 0.000000e+00 | 2 0 1 0 |
| 18 | 0.000000e+00 | 2 0 0 1 |
| 19 | 0.000000e+00 | 1 2 0 0 |
| 20 | 0.000000e+00 | 1 1 1 0 |
| 21 | 0.000000e+00 | 1 1 0 1 |
| 22 | -4.703013e+00 | 1 0 2 0 |
| 23 | -1.778965e+00 | 1 0 1 1 |
| 24 | 5.331860e+01 | 1 0 0 2 |
| 25 | 0.000000e+00 | 0 3 0 0 |
| 26 | 0.000000e+00 | 0 2 1 0 |
| 27 | 0.000000e+00 | 0 2 0 1 |
| 28 | 4.937315e+00 | 0 1 2 0 |
| 29 | 2.033959e+02 | 0 1 1 1 |
| 30 | -5.597491e+01 | 0 1 0 2 |

**All information - Tori profile, frequency and stability are hidden in AIs!**

# Phase Space at Mid-Plane

- Can't visualize in 4D space, so let's project to 2D mid-plane

$$\mathcal{K} = \sum_{ab} c_{ab} x^a p_x^b$$




Red lines: approximate invariant contours
If closed (torus), motion is stable
Outermost torus is **dynamic aperture boundary**

Black dots: simulated stable motion

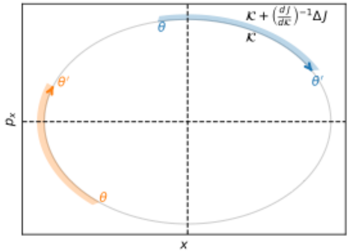
Blue dots: simulated unstable motion

Mid-plane with $y = p_y = 0$

# Betatron Tune’s Fractional Part

## Poincaré Rotation Number (PRN)

Poincaré map

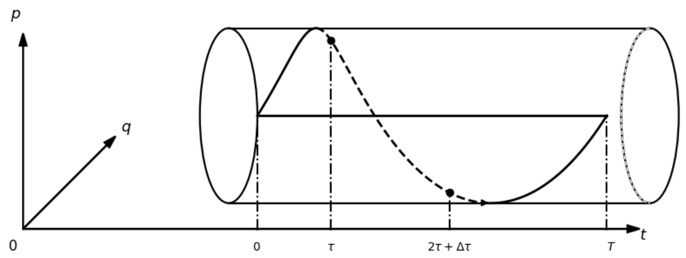



$$\nu_i = \frac{\int_x^{x'} \left(\frac{\partial \mathcal{K}}{\partial p_x}\right)^{-1} dx}{\oint \left(\frac{\partial \mathcal{K}}{\partial p_x}\right)^{-1} dx} = \frac{dJ'}{d\mathcal{K}} \Big/ \frac{dJ}{d\mathcal{K}} \approx \frac{\Delta J'}{\Delta \mathcal{K}} \Big/ \frac{\Delta J}{\Delta \mathcal{K}}$$

$S = 2\pi J \text{ and } J\text{: action}$

**S. Nagaitsev and T. Zolkin, Phys. Rev. Accel. Beams 23, 054001**
**C. Mitchell, R. Ryne, K. Hwang, S. Nagaitsev and T. Zolkin, Phys. Rev. E 103, 062216**

## Time-of-Flight of effective Hamiltonian



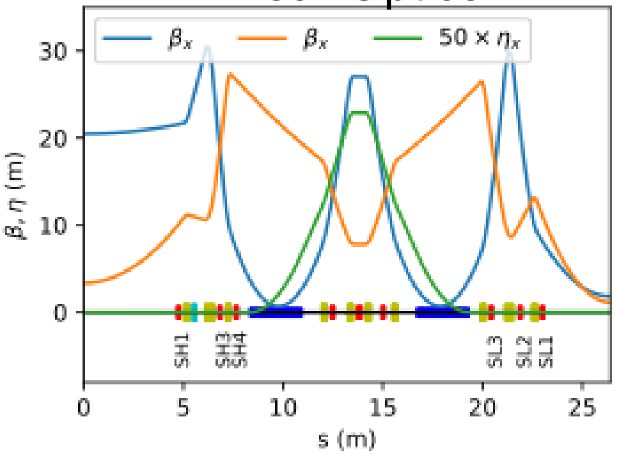

$$\nu_i = \frac{\tau}{T}$$

D. Xu, Y. Li, Y. Hao and S. Nagaitsev
arXiv:2512.16060

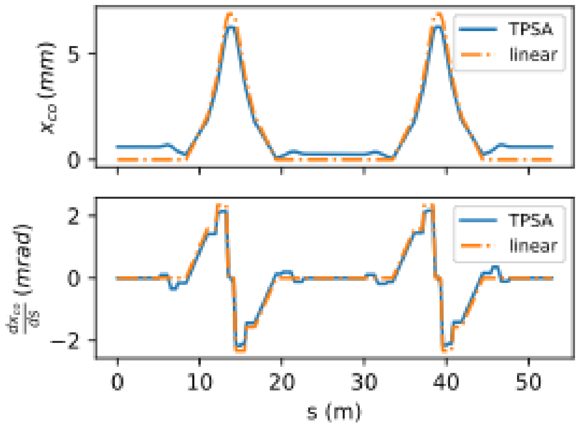
**Quasiperiodic** tune $$\nu = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \nu_i$$

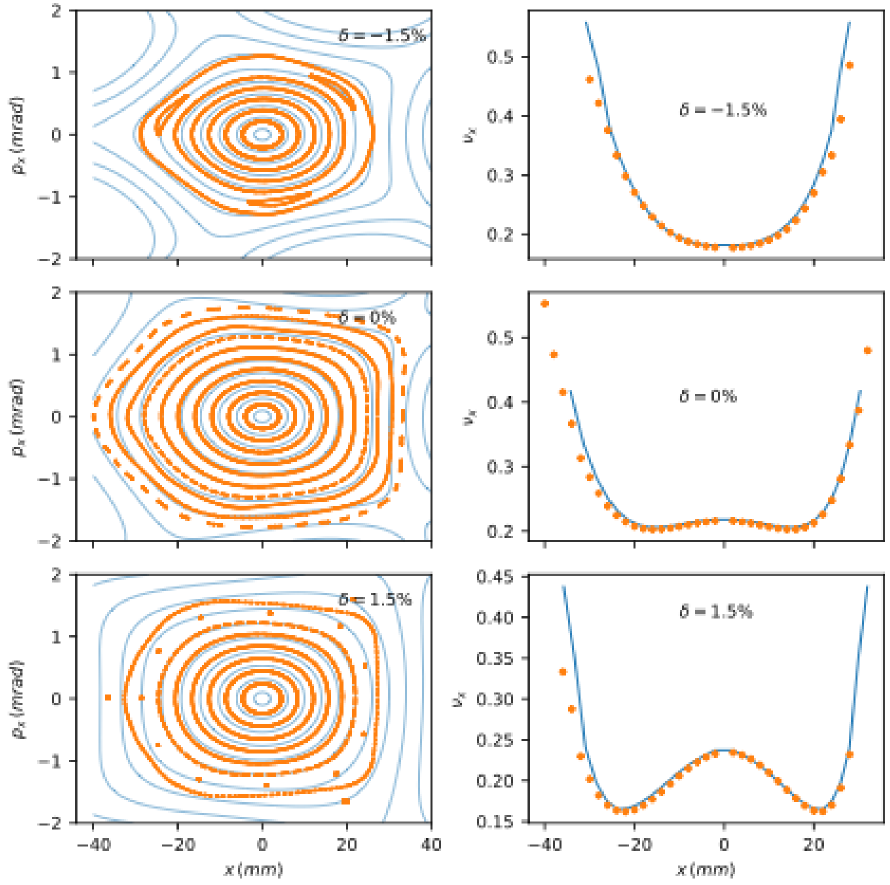
# Amplitude-Dependent Detuning (ADD)

Linear Optics




Dispersive Orbit



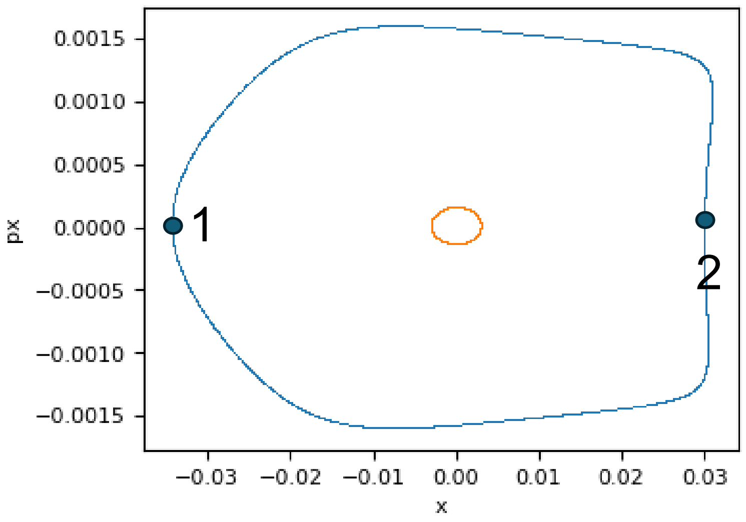

Both DAs and ADDs reasonably agree with tracking simulation

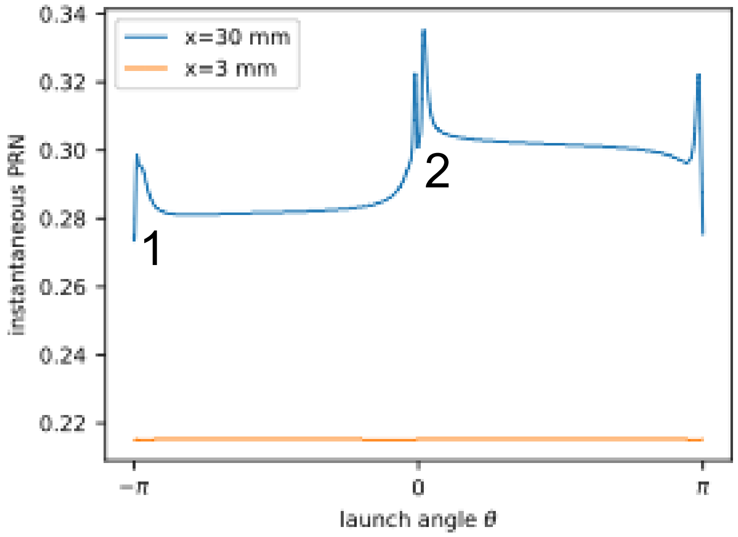
Blue lines: AIA

Yellow dots: tracking simulation

# Tune diffusion

Tune diffusion: **Sensitivity** on initial condition



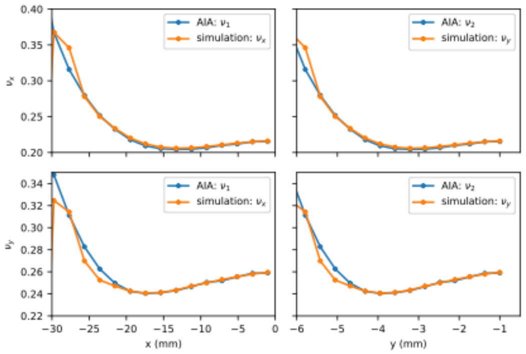



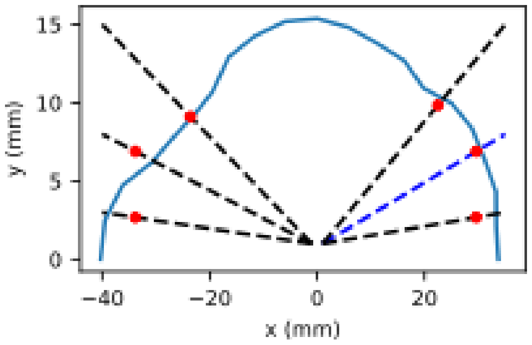

Once quasi-periodicity is lost, the motion become unstable.

# Summary

- A new nonlinear beam dynamics framework
- Most existing mathematical tools, such as Hamiltonian perturbation, canonical transformation, Lie algebra technique, normal form, even Courant-Synder parameterization etc. are **NOT** needed
- Dynamic aperture and fractional tune can be estimated quite accurately
- Although only 1DoF is demonstrated here, we already extend it to 2DoF

Tune-shift-with-amplitude




Dynamic aperture

# Acknowledgements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Questions?

AIA
Approximate Invariant
ANALYSIS

LOGO

# Back-up: resonance, inversibility

Cubic Henon map $\quad q' = p, \qquad p' = -q + p + p^3$

$$V^{(i+1)} = (I - m_{22})^{-1} m_{21} V^{(i)}$$

$$M^{\mathrm{T}} = \left[\begin{array}{cc|ccc|cccc|ccccc}
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & -3 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -4 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 3 & 6 \\
0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & -1 & -2 & -3 & -4 \\
0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1
\end{array}\right]$$

Rank deficient matrices

because $\quad \mu = \dfrac{\pi}{3}$

$$\mathrm{K_B}(q, p) = q^2 - qp + p^2 - q^2p^2 + \frac{7}{15}\left(q^2 - qp + p^2\right)^2$$

# Push to higher order?

Adding higher-order kick terms changes the projected driving terms:

$$q' = p, \qquad p' = -q + p + p^3 + p^5, \qquad \left(I - m_{66}^{\mathrm{T}}\right) v_6 = m_{26}^{\mathrm{T}} v_2 + m_{46}^{\mathrm{T}} v_4$$

$$\Longrightarrow \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 1 & 0 & -1 & -5 & -15 \\ 0 & 0 & 0 & 2 & 4 & 10 & 20 \\ 0 & 0 & -1 & -3 & -5 & -10 & -15 \\ 0 & 1 & 2 & 3 & 4 & 6 & 6 \\ -1 & -1 & -1 & -1 & -1 & -1 & 0 \end{bmatrix} v_6 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -4B \\ 6B \\ -6B \\ 2B \end{bmatrix}$$

$$\Longrightarrow \qquad v_6 = \frac{2B}{17} (1, -3, -4, -4, -4, -3, 1)^{\mathrm{T}} + \mathrm{Null}\left(I - m_{66}^{\mathrm{T}}\right)$$

Ignoring the freedom of null space, invariants upto 6th order:

$$\mathcal{K} = A\left(q^2 - qp + p^2 - q^2p^2\right) + B\left(q^2 - qp + p^2\right)^2 + \frac{2B}{17}\left(q^6 - 3q^5p - 4q^4p^2 - 4q^3p^3 - 4q^2p^4 - 3qp^5 + p^6\right)$$

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