THE DETECTION OF THE FIXED LINES IN FOUR-DIMENSIONAL PHASE SPACE *

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Abstract

One-dimensional resonances create fixed points and islands in the Two-dimensional phase space, which has long been recognized as crucial for the diffusion of particles in high-intensity bunches. Coupled resonances are even more relevant, but their dynamics is more difficult to grasp as it happens in a four-dimensional phase space. Following a conceptual and theoretical investigation, a detailed experimental campaign has been carried out in the CERN SPS to investigate the existence of the four-dimensional fixed lines. The experimental investigation has been exceptional complex as it required an extreme control of all accelerator parameters. We report here the difficulties encountered and the main findings, their impact, and possible applications.

INTRODUCTION

Nonlinear resonances are of concern for high-intensity and high brightness beams, as for the SIS100 in the Facility for Antiproton and Ion Research (FAIR) project at GSI [1], and for the operation of the accelerator chain at CERN after the LHC injectors upgrade (LIU) [2]. Studies performed over the last 15 years on one-dimensional (1D) resonances have shown that space charge induced periodic resonance crossing is a prominent mechanism behind halo formation and associated particle loss for high-intensity bunches.

1D-RESONANCES AND SPACE CHARGE

The presence of nonlinear magnetic elements in a synchrotron excites resonances that modify the topology of the orbits in the Poincaré surface of section. The resulting *resonant dynamics* of one-dimensional resonances (of the form $n_xQ_x = N$ or $n_yQ_y = N$) is characterized by distinctive structures in the Poincaré surface of section, namely fixed points, islands, and separatrices. In this case, the two degrees of freedom are decoupled, and the mixed coordinate planes (x, y) and (x', y') exhibit the characteristic rectangular structure described in Refs. [3, 4]. Islands generated by one-dimensional resonances have been observed in several facilities, including CERN, the Tevatron, and the Indiana University Cyclotron Facility [4–6]. Precise control of these islands is essential for the implementation of novel extraction schemes (see Refs. [7–9]).

In a bunched beam, the slow synchrotron motion drives particles through regions of varying longitudinal bunch density, thereby inducing a nonlinear modulation of the transverse tune due to space charge effects. Because the synchrotron motion is slow, the transverse dynamics operates in the regime of "amplitude modulation," as described in Ref. [10]. In this regime, the transverse motion of a particle experiences a modulation of the amplitude-dependent space-charge tune shift. This effect becomes particularly relevant when a lattice one-dimensional resonance overlaps with the space-charge tune spread of the bunched beam. Under these conditions, periodic resonance crossing can occur, driven by the interplay between transverse incoherent space charge and synchrotron motion. Experiments performed at the CERN-PS [11] and at the SIS18 at GSI [12] have confirmed this mechanism.

2D-RESONANCES AND SPACE CHARGE

For two-dimensional (2D) resonances, the orbits in the Poincaré surface of section evolve in a four-dimensional phase space whose topology may elude direct geometric intuition. When the accelerator tunes are set near the third-order resonance, i.e. when the distance to the resonance satisfies $\Delta_r = Q_x + 2Q_y - N \simeq 0$, the particle dynamics acquire distinctive features due to the presence of nonlinear fields. In particular, the phase advances per turn, $\Delta\phi_x$ and $\Delta\phi_y$, are no longer constant, and the single-particle emittances ϵ_x and ϵ_y, computed from the particle coordinates according to the definition of Courant and Snyder [13], are no longer invariant; we therefore denote them by a_x and a_y. A perturbative analysis of the dynamics shows that a_x and a_y satisfy the relation $2a_x = a_y + C$, where C is a constant determined by the initial conditions. Consequently, the resonance properties can be described using only a_y, together with the phase combination $\Omega = \phi_x + 2\phi_y$. For a fixed value of C, the pair (Ω, a_y) evolves turn by turn in this space. However, there exists a stationary pair of values $(\Omega_{\text{fl}}, a_y)$ [14]. When transformed back to the Courant-Snyder coordinates $(\hat{x}, \hat{p}_x, \hat{y}, \hat{p}_y)$ at a given accelerator location, i.e. on the Poincaré surface of section, this special solution acquires the topology of a one-dimensional closed curve parameterized as follows:

$$\begin{aligned} \hat{x}(t) &= \sqrt{a_x}\cos(-2t - \alpha + \pi M), \\ \hat{y}(t) &= \sqrt{a_y}\cos(t), \\ \hat{p}_x(t) &= -\sqrt{a_x}\sin(-2t - \alpha + \pi M), \\ \hat{p}_y(t) &= -\sqrt{a_y}\sin(t). \end{aligned} \tag{1}$$

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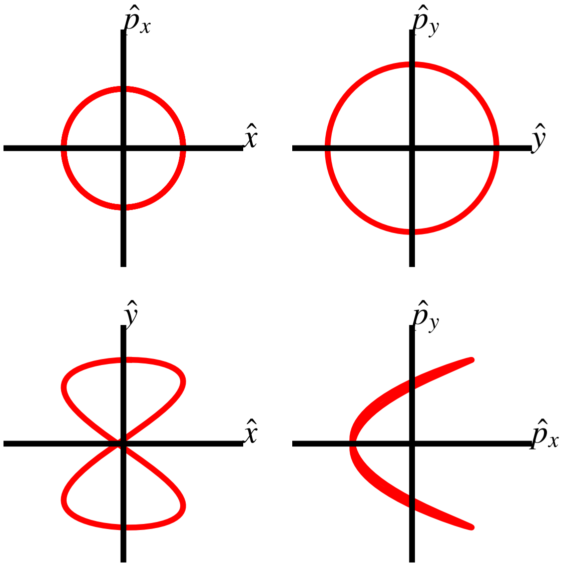

Figure 1: Poincaré surface of section for 2D third order coupled resonance. This picture shows the characteristics of the projections in the main phase space planes. From Ref. [15].

Here, t is a parameterization variable with $0 < t \leq 2\pi$, while a_x and a_y are stationary quantities. The parameter α denotes the angle of the resonance driving term with respect to the Poincaré surface of section, and the integer M takes the value 0 or 1 depending on the signs of Δ_r and α. From Eq. (1), one obtains the stationary phase advance of this resonant four-dimensional structure, $\Omega_{\text{fl}} = -\alpha + \pi M$, which characterizes its geometric "orientation" in phase space. The resonant orbit described by Eq. (1) is illustrated in Fig. 1. The figure shows that each individual (q, p) plane associated with a single degree of freedom appears essentially unaffected, so that the signature of the resonant dynamics becomes visible only in the mixed planes. The four panels in Fig. 1 display projections of the *four-dimensional closed curve* defined by Eq. (1). This structure, referred to as a "*fixed line*" [16], has the property that any particle belonging to it, is located somewhere along the curve at each passage through the Poincaré surface of section [14].

For the storage of an intense bunch, the same mechanism observed for one-dimensional resonances is expected to occur in a similar manner, although with the additional complexity of halo diffusion developing in the four-dimensional phase space. Experiments at the CERN-PS with intense bunched beams, for which the space-charge tune spread overlaps a third-order coupled resonance [17], have revealed the formation of an asymmetric halo, as shown in Fig. 2. Figure 3 (top) shows the simulated space-charge tune spread together with the third-order resonance. The two circles (black and red) correspond to the simulated orbits in the x–y Poincaré section displayed in Fig. 3 (bottom). These orbits exhibit shapes similar to those shown in Fig. 1 (bottom left).

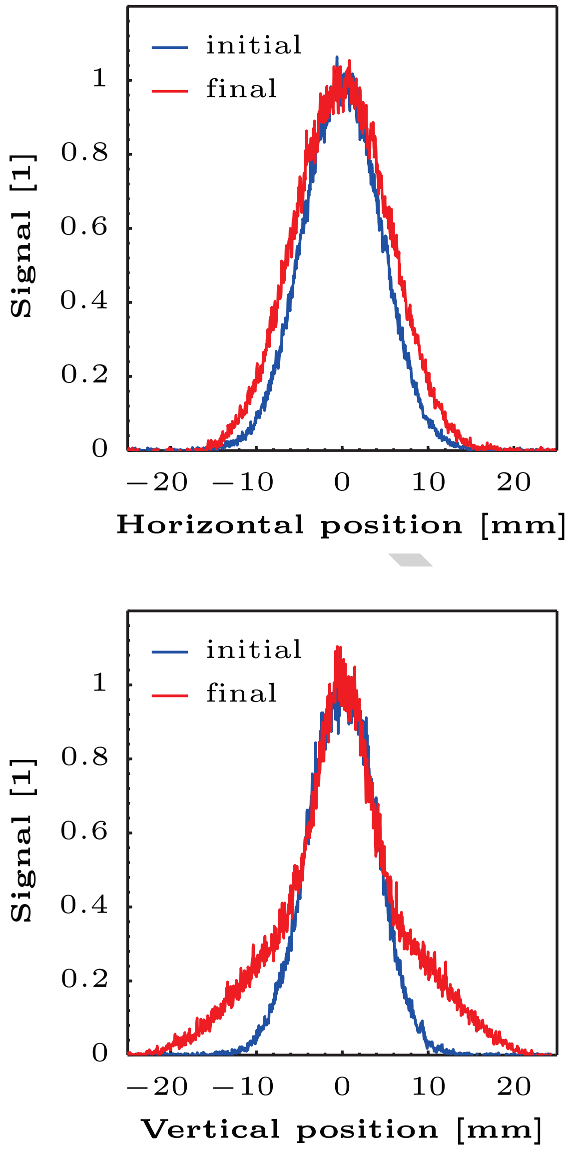

Figure 2: From Ref. [17], experimental beam profiles at the beginning/end of the beam storage in the horizontal plane (top) and the vertical plane (bottom) for a machine working point for which the space-charge tune spread overlaps a third-order coupled resonance.

This numerical result suggested that the geometry of the fixed lines is responsible for the observed halo asymmetry.

Despite these extensive theoretical and numerical studies, the existence of fixed lines had not been experimentally demonstrated at the time the studies reported in Ref. [17] were carried out. This absence of experimental evidence was unsatisfactory, since fixed lines were regarded as a key element of the mechanism underlying periodic resonance crossing and the resulting asymmetric halo formation. This situation motivated the experimental verification of fixed lines.

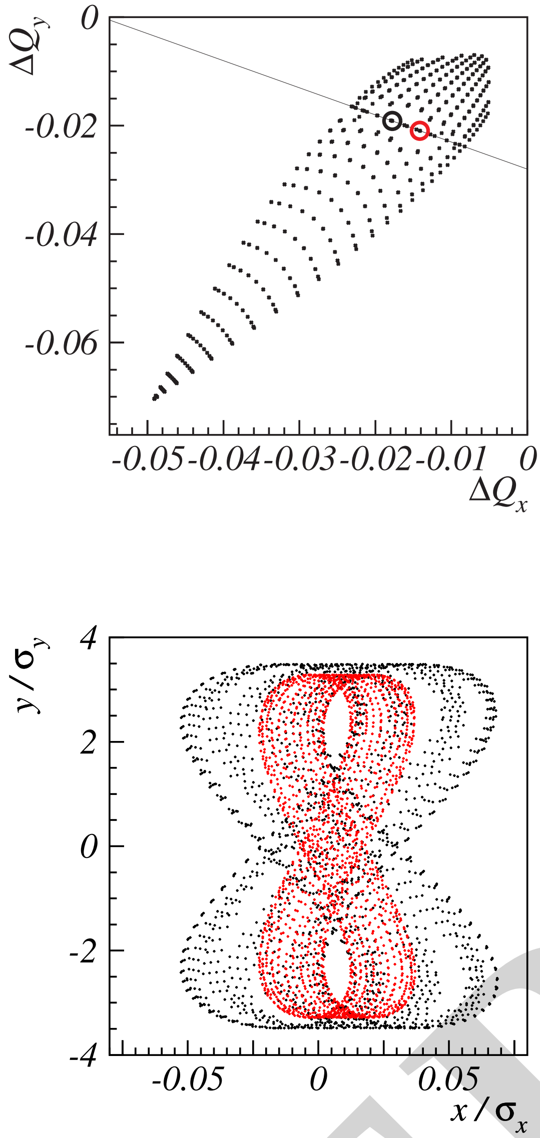

Figure 3: From Ref. [17] Simulations of the space charge tunespread (top), and two resonant orbits at $z = 0$ (bottom).

THE EXPERIMENT AND SET UP

The measurement of fixed lines was performed at the CERN SPS, taking advantage of the small transverse size of a high-energy pencil beam. Transverse oscillations were excited with kicker magnets, while a third-order resonance was driven by strongly powered sextupoles. Beam positions were recorded turn by turn using the Beam Position Monitors (BPMs). Since consecutive BPMs in each plane are separated by a phase advance of about 90°, the Courant–Snyder coordinates $(\hat{x}, \hat{p}_x, \hat{y}, \hat{p}_y)$ could be reconstructed at a chosen machine location. The goal of the campaign was to kick the beam onto a fixed line and observe the associated phase-space structure, but the experiment presented three main challenges. First, the effect is intrinsically fragile: tune modulation caused by power-converter ripple and magnetic-field fluctuations can perturb the conditions required for fixed-line observation. To mitigate this, the beam was accelerated to 100 GeV/c before the transverse excitation, and the machine settings were carefully optimized. Second, quadrupole manufacturing tolerances generate “beta beating,” an optics perturbation of up to ~ 5%, which had to be included in the analysis. Third, the SPS provides only one horizontal and one vertical kicker suitable for the experiment, restricting the accessible fixed-line orientation to a single resonance phase $\Omega = \Omega_u$. In addition, the synchronization of the kickers had to be carefully controlled (see Ref. [15]). Because of the difficulties listed above, a sequence of measurements was performed by systematically scanning the sextupole strengths $K_{2,1}, K_{2,2}$, the distance from resonance Δ_r, and the kicker amplitudes θ_x, θ_y. The beam can be transferred onto a fixed line only if the angle α of the driving term is consistent with the combined effect of the kickers and the lattice between them. Since the SPS lacks a single sextupole at the required phase advance, fixed lines were searched for by scanning α to identify suitable conditions. Once evidence of a fixed line was found, scans of Δ_r, sextupole strengths, and kicker amplitudes were carried out. For every measurement, turn-by-turn beam positions from all available BPMs were recorded.

DETECTION OF THE FIXED LINES

Figure 4 shows a measured Poincaré surface of section. The blue markers represent scaled beam coordinates (see Methods in Ref. [15]) obtained experimentally from 3000 passages through a selected longitudinal observation point. **(a)** Projection onto the horizontal plane $(\bar{x}, \bar{p}_x)$ in scaled

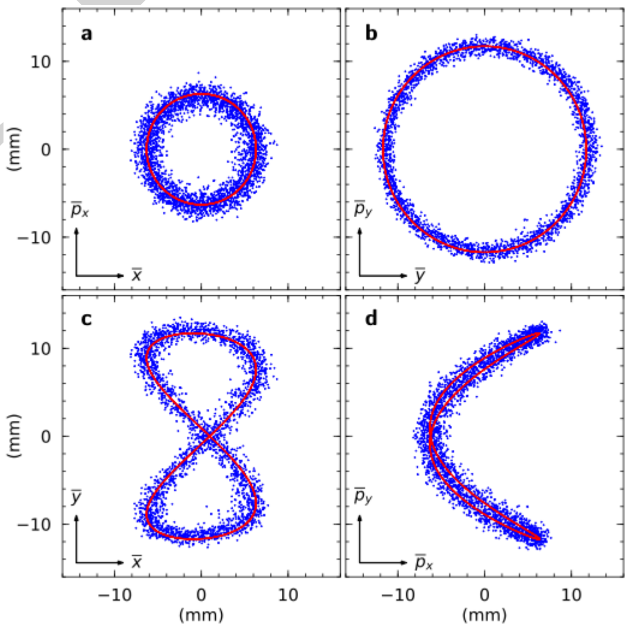

Figure 4: Experimental measurement of the fixed lines in scaled Courant-Snyder coordinates. Picture from Ref. [15].

Courant–Snyder coordinates. The circular shape indicates that the amplitude $\bar{a}_x$ is constant. **(b)** Projection onto the vertical plane $(\bar{y}, \bar{p}_y)$, where the circular orbit indicates that $\bar{a}_y$ is constant. **(c)** and **(d)** show the mixed-coordinate projections $(\bar{x}, \bar{y})$ and $(\bar{p}_x, \bar{p}_y)$, respectively, where the motion follows a Lissajous curve. The red line corresponds to the best fit of Eq. (1) to the experimental data, confirming that the observed topology is "consistent" with that of a fixed line. However, these results alone do not prove that the orbit in Fig. 4 is resonant. The same phase-space structure would be observed even in the absence of resonance excitation, since linear dynamics with $Q_x + 2Q_y = N$ also produces these Lissajous curves. To demonstrate that Fig. 4 corresponds to a fixed line, it is necessary to show that the associated (Ω, a_y) constitutes a stationary point.

Using the action–angle representation $\hat{x} = \sqrt{a_x}\cos\phi_x$, $\hat{p}_x = -\sqrt{a_x}\sin\phi_x$, $\hat{y} = \sqrt{a_y}\cos\phi_y$, $\hat{p}_y = -\sqrt{a_y}\sin\phi_y$, one can reconstruct a_x, a_y, ϕ_x, ϕ_y, and Ω. This allows the Poincaré surface of section to be represented in Courant–Snyder variables and the resonant dynamics to be examined in the (Ω, a_y) space. If the beam is locked to a stable fixed line, both Ω and a_y should remain constant. In Fig. 5(a), we show the transformation of the data from Fig. 4 into (Ω, a_y). In these coordinates, the trajectory rotates around a fixed point. The observed oscillations indicate that the beam was not placed exactly on the fixed line, but with a slight offset, leading to beam motion around a fixed line. This observation provides strong evidence that Fig. 4 corresponds to the projection of the beam motion near a fixed line. In Fig. 5(b), we also show a measurement affected by a drift in the accelerator parameters. The resulting spiral behavior reflects a gradual shift of the oscillation center, consistent with trapping the SPS beam around a slowly moving fixed line in the four-dimensional phase space.

OUTLOOK

The relevance of this experimental campaign goes beyond a proof-of-principle demonstration. The observation and confirmation of the main features predicted by numerical simulations and analytical theories suggest that the phenomenon of coupled-resonance periodic crossing deserves full consideration.

In Fig. 6, the effect of periodic crossing induced by the interplay between space charge and synchrotron motion shows a diffusion process. The preservation of the invari-

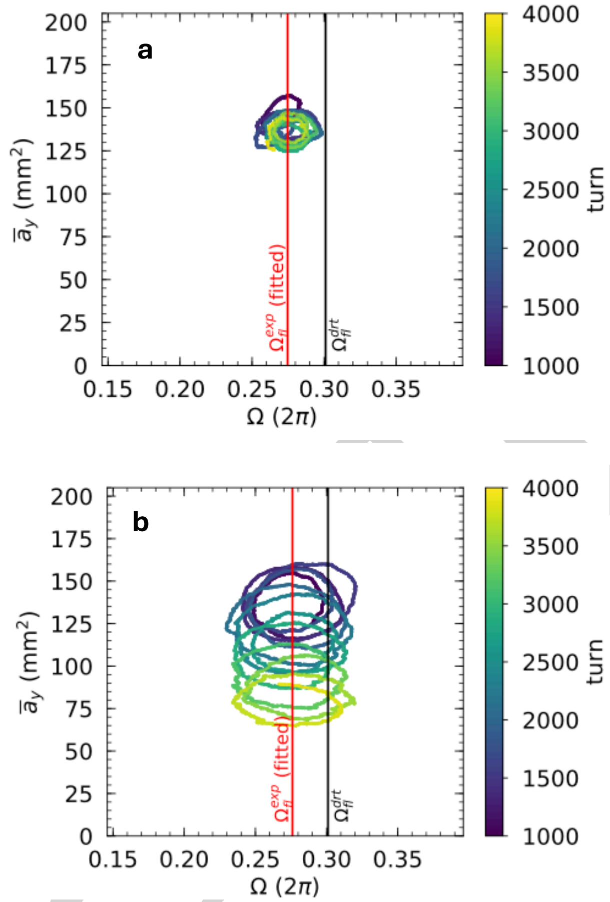

Figure 5: Top: Oscillation of (Ω, a_y) form the data of Fig. 3. Bottom: Trapping of the beam onto the "moving" fixed line. Here a_y is scaled to $\bar{a}_y$. Picture from Ref. [15], see also Methods.

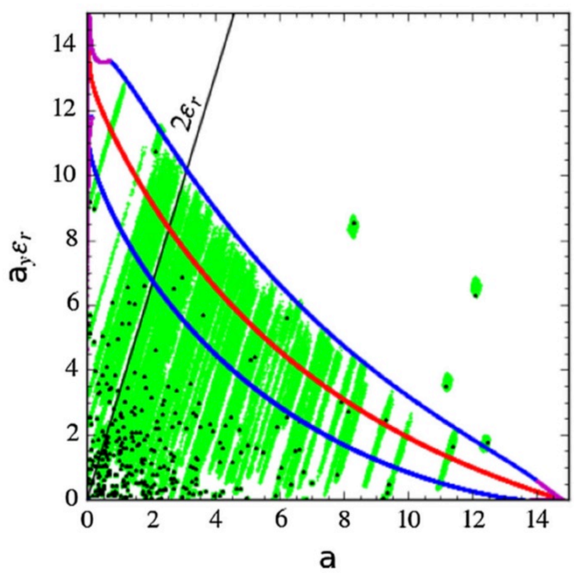

Figure 6: Diffusion of particles induced by periodic crossings with fixed lines. Note that diffusion occurs only for particles that cross the fixed lines. Figure adapted from Ref. [18]. The red line represents the collection of all the scaled fixed lines at the location of maximum space charge in a high-intensity bunched beam. The blue lines indicate the outer and inner edges of the separatrix at the same longitudinal location. The green traces show the direction of diffusion of particles during storage because of the periodic crossing of the fixed lines. The black line indicates the direction of the invariant $2a_x = a_y + C$ for the case of $C = 0$; for more details on the scaled quantities in this picture see in Ref. [18].

ant $2a_x = a_y + C$, and the associated diffusion of particles along this invariant, allows the determination of the halo extent. This halo growth is directly linked to the geometry of the fixed lines in phase space. An advanced application also consists in using fixed lines to generate beam correlations for studying how these correlations survives along the accelerator chain, as discussed in Ref. [19].

REFERENCES

[1] P. Spiller and G. Franchetti, "The FAIR accelerator project at GSI", *Nucl. Instrum. Methods Phys. Res. A*, vol. 561, no. 2, pp. 305–309, Jun. 2006. doi:10.1016/j.nima.2006.01.043

[2] H. Damerau *et al.*, "LHC injectors upgrade, technical design report", CERN, Geneva, Switzerland, Rep. CERN-ACC-2014-0337, Dec. 2014. https://cds.cern.ch/record/1976692

[3] J. Binney and S. Tremaine, *Galactic dynamics, 2nd Ed.* Princeton, NJ, USA: Princeton University Press, 2008.

[4] A. Chao *et al.*, "Experimental investigation of nonlinear dynamics in the fermilab tevatron", *Phys. Rev. Lett.*, vol. 61, no. 24, pp. 2752–2755, Dec. 1988. doi:10.1103/PhysRevLett.61.2752

[5] S. Y. Lee *et al.*, "Experimental determination of a nonlinear Hamiltonian in a synchrotron", *Phys. Rev. Lett.*, vol. 67, no. 27, pp. 3768–3771, Dec. 1991. doi:10.1103/PhysRevLett.67.3768

[6] M. Giovannozzi and P. Scaramuzzi, "Nonlinear dynamics studies at the CERN proton synchrotron: precise measurements of islands parameters for the novel multi-turn extraction", in *Proc. EPAC'04*, Lucerne, Switzerland, paper WEPLT018, pp. 1861–1863, Sep. 2004. http://accelconf.web.cern.ch/e04/papers/WEPLT018.pdf

[7] R. Cappi and M. Giovannozzi, "Multiturn extraction and injection by means of adiabatic capture in stable islands of phase space", *Phys. Rev. ST Accel. Beams*, vol. 7, no. 2, p. 024001, Feb. 2004. doi:10.1103/PhysRevSTAB.7.024001

[8] S. Gilardoni *et al.*, "Experimental evidence of adiabatic splitting of charged particle beams using stable islands of transverse phase space", *Phys. Rev. ST Accel. Beams*, vol. 9, no. 10, p. 104001, Oct. 2006. doi:10.1103/PhysRevSTAB.9.104001

[9] D. E. Veres *et al.*, "Proof-of-principle experiment of a novel beam extraction over millions of turns using stable resonance islands and bent crystal", *Phys. Rev. Res.*, May 2026. doi:10.1103/n93z-3ry8

[10] T. Satogata *et al.*, "Driven response of a trapped particle beam", *Phys. Rev. Lett.*, vol. 68, no. 12, pp. 1838–1841, Mar. 1992. doi:10.1103/PhysRevLett.68.1838

[11] G. Franchetti, I. Hofmann, M. Giovannozzi, M. Martini, and E. Metral, "Space charge and octupole driven resonance trapping observed at the CERN proton synchrotron", *Phys. Rev. ST Accel. Beams*, vol. 6, no. 12, p. 124201, Dec. 2003. doi:10.1103/PhysRevSTAB.6.124201

[12] G. Franchetti *et al.*, "Experiment on space charge driven nonlinear resonance crossing in an ion synchrotron", *Phys. Rev. ST Accel. Beams*, vol. 13, no. 11, p. 114203, Nov. 2010. doi:10.1103/PhysRevSTAB.13.114203

[13] ED. Courant and HS. Snyder, "Theory of the alternating gradient synchrotron", *Annals Phys.*, vol. 3, pp. 1–48, 1958. doi:10.1016/0003-4916(58)90012-5

[14] G. Franchetti and F. Schmidt, "Extending the nonlinear-beam-dynamics concept of 1d fixed points to 2d fixed lines", *Phys. Rev. Lett.*, vol. 114, no. 23, p. 234801, Jun. 2015. doi:10.1103/PhysRevLett.114.234801

[15] H. Bartosik, G. Franchetti, and F. Schmidt, "Observation of fixed lines induced by a nonlinear resonance in the CERN super proton synchrotron", *Nat. Phys.*, vol. 20, no. 6, pp. 928–933, 2024. doi:10.1038/s41567-023-02338-3

[16] F. Schmidt, "Untersuchungen zur dynamischen Akzeptanz von Protonenbeschleunigern und ihre Begrenzung durch chaotische Bewegung", Ph.D. thesis, Hamburg Univ., Hamburg, Germany, 1988.

[17] G. Franchetti, S. Gilardoni, A. Huschauer, F. Schmidt, and R. Wasef, "Space charge effects on the third order coupled resonance", *Phys. Rev. Accel. Beams*, vol. 20, no. 8, p. 081006, Aug. 2017. doi:10.1103/PhysRevAccelBeams.20.081006

[18] G. Franchetti, "Incoherent effects of space charge and sum resonances on particle beams in a storage ring", *Phys. Rev. Accel. Beams*, vol. 22, no. 11, p. 114201, Nov. 2019. doi:10.1103/PhysRevAccelBeams.22.114201

[19] E. Lamb *et al.*, "Measurements of beam correlations induced via coupled resonance crossing in the CERN PSB", in *Proc. IPAC'24*, Nashville, TN, USA, pp. 1006–1009, May 2024. doi:10.18429/JACoW-IPAC2024-TUPC07