IPAC23

Physics of Circular Accelerators/Colliders

Eliana Gianfelice-Wendt
(Fermilab)

Contents:

- The basics of (modern) accelerator rings
  - From Lorentz force to Hill's equation
- Basic concepts in Particle Accelerators
  - Closed Orbit, Chromaticity, Dispersion
- Rings as particle colliders

Remarks:

- It is assumed that particles have large momentum and the ring is large.
- Only *single* particle dynamics (no collective effects).
- Not covered
  - Synchrotron Radiation.
  - Longitudinal dynamics.
  - ...and much more!

The very basic concepts

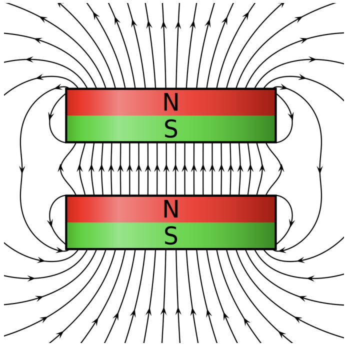
Charged particles in EM fields experience the **Lorentz force**:

$$\frac{d\vec{p}}{p} = \vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

- The magnetic force is always perpendicular to the velocity: it changes the trajectory direction.
- The electric field may deflect and/or change the momentum.

Building blocks of a high energy accelerator:

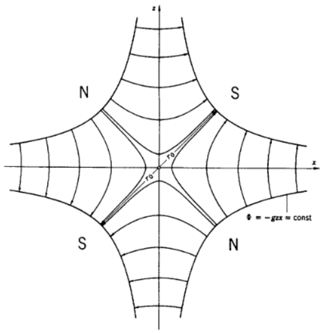
- Dipole magnets (uniform field) define the *design trajectory*, closed in a storage ring.



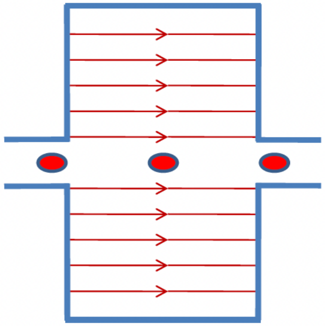
- "Normal" quadrupole magnets aligned along the design trajectory

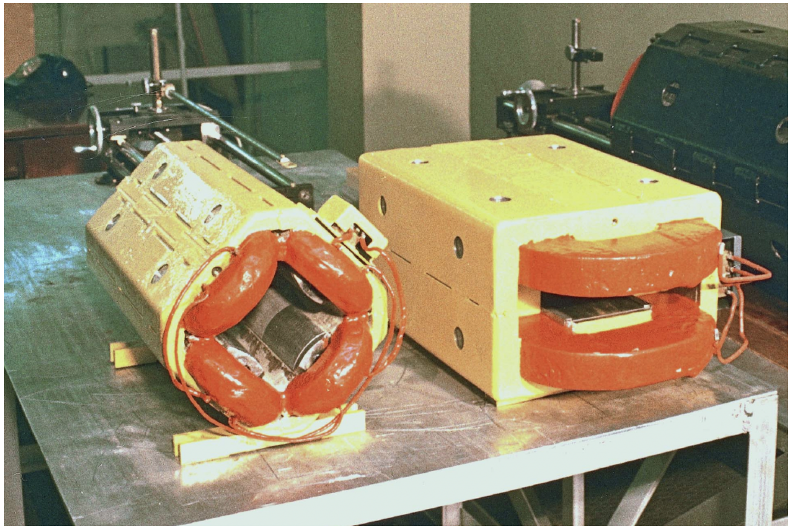
$$B_x = gy \qquad B_y = gx$$

bend horizontally particles with horizontal offset and vertically those with vertical offset: provide *transverse focusing*.

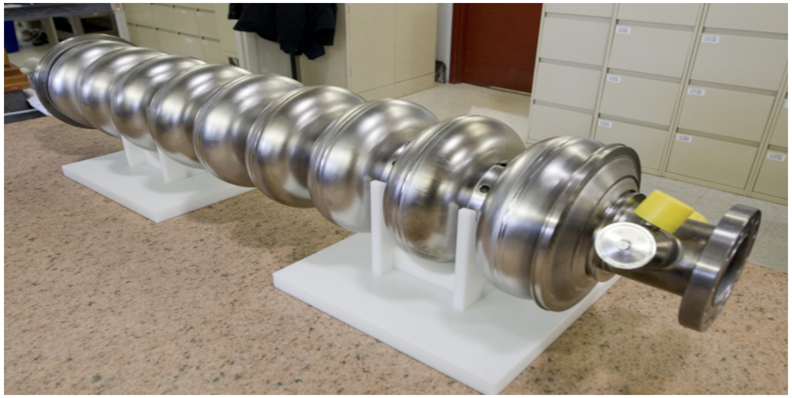


- RF cavities with electric field parallel to the velocity (de)accelerate and/or provide *longitudinal focusing*.



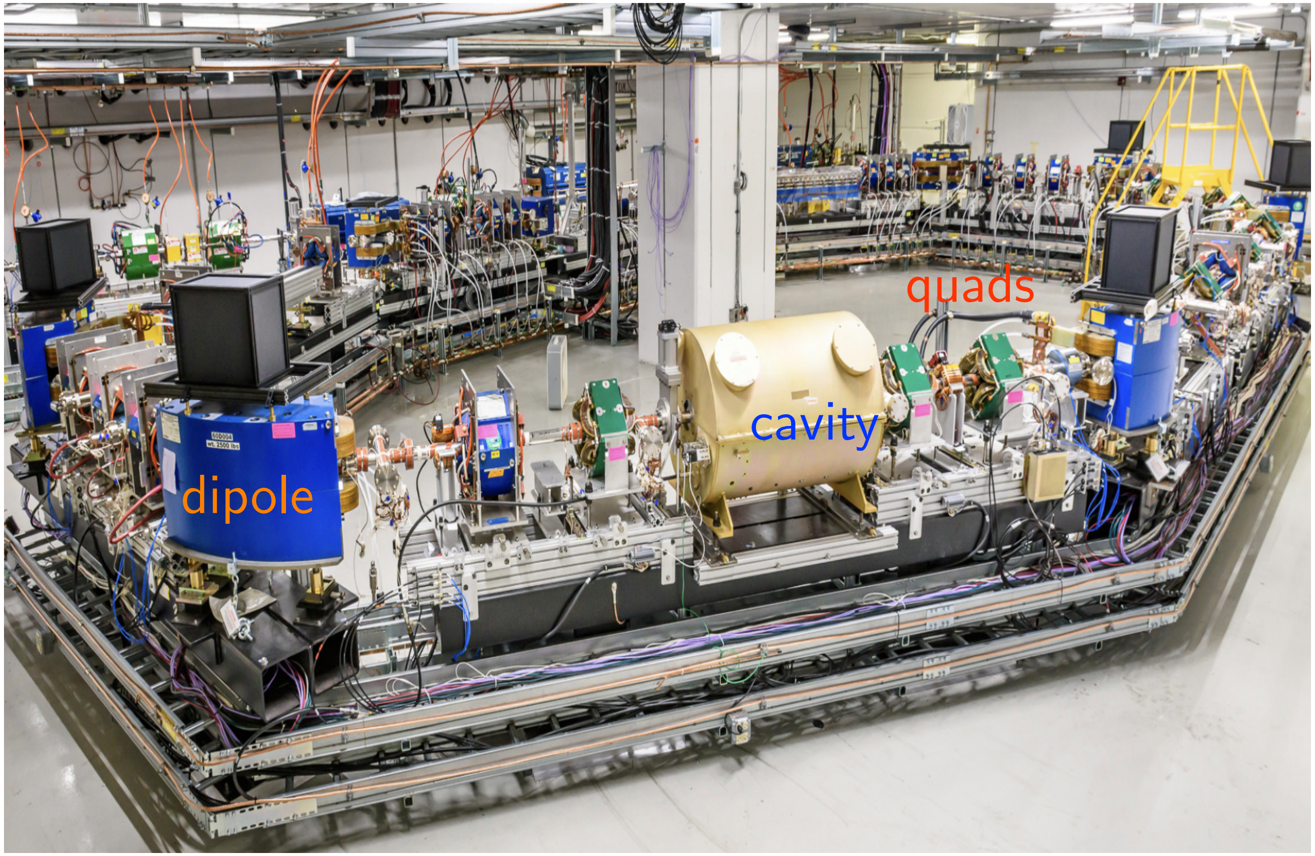


A quadrupole and a dipole.



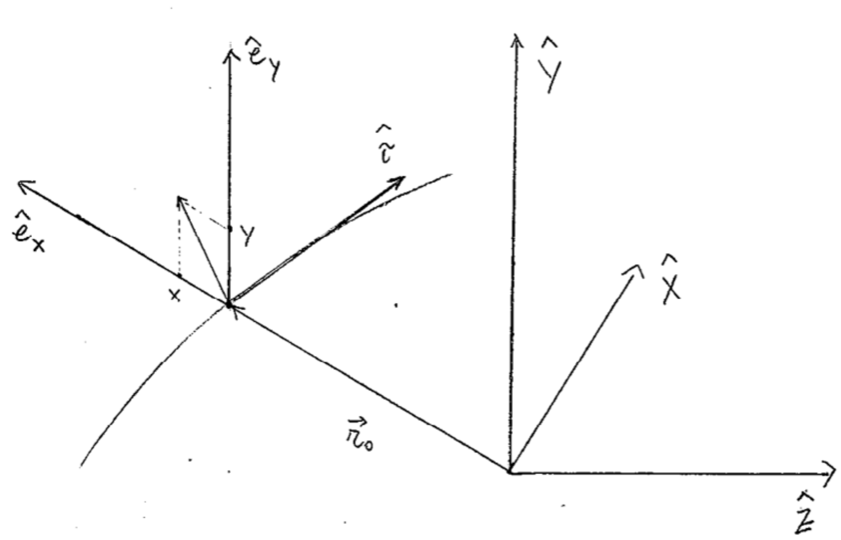
9 cell SC RF cavity

Fermilab IOTA ring: a 40 m long test facility.



From basic concepts to formulas

It is convenient to describe the particle motion in a right-handed rectangular system $(\hat{\boldsymbol{\tau}}, \hat{\boldsymbol{e}}_{\boldsymbol{x}}, \hat{\boldsymbol{e}}_{\boldsymbol{y}})$ which moves along the nominal trajectory. In such a frame and using the path length $\boldsymbol{s}$ rather than the time $\boldsymbol{t}$ the motion is described by **Hill's equations**.



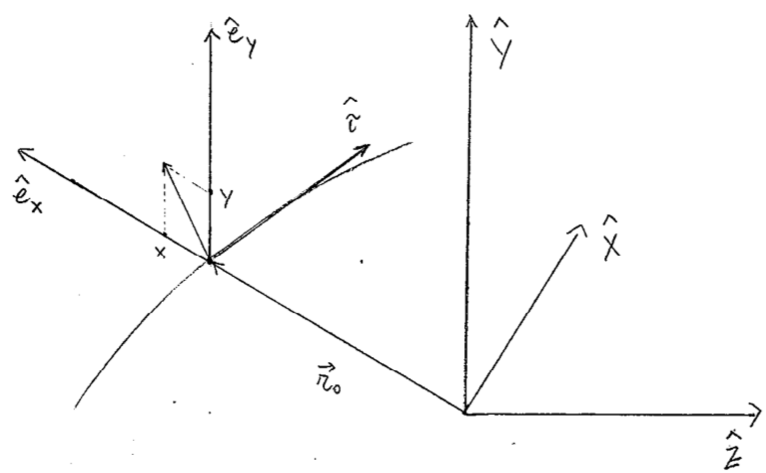
Steps:

- Start with the equations of motion under Lorentz force in the laboratory coordinates system

$$\frac{\boldsymbol{d}}{\boldsymbol{dt}}\left[\boldsymbol{m}\frac{\boldsymbol{d\vec{r}}}{\boldsymbol{dt}}\right] = \boldsymbol{e}\frac{\boldsymbol{d\vec{r}}}{\boldsymbol{dt}} \times \vec{\boldsymbol{B}}$$

- Use of the *local* coordinate system attached to the reference orbit (assumed here for simplicity planar and lying in the horizontal plane):

$$\begin{cases} \hat{\boldsymbol{\tau}} \text{ tangent to the orbit} \\ \hat{\boldsymbol{e}}_{\boldsymbol{y}} \perp \text{ to the plane of the orbit (constant)} \\ \hat{\boldsymbol{e}}_{\boldsymbol{x}} \equiv \hat{\boldsymbol{e}}_{\boldsymbol{y}} \times \hat{\boldsymbol{\tau}} \end{cases}$$



- Use Frenet formulas

$$\begin{cases} \frac{d}{ds}\hat{\boldsymbol{e}}_{\boldsymbol{x}} = \hat{\boldsymbol{\tau}}\frac{1}{\rho} \\ \frac{d}{ds}\hat{\boldsymbol{\tau}} = -\hat{\boldsymbol{e}}_{\boldsymbol{x}}\frac{1}{\rho} \end{cases}$$

- Express particle position in terms of the reference particle position vector $\vec{\boldsymbol{r}}_0$

$$\vec{\boldsymbol{r}}(\boldsymbol{s},\boldsymbol{x},\boldsymbol{y}) \equiv \vec{\boldsymbol{r}}_0(\boldsymbol{s}) + \boldsymbol{d}\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_0(\boldsymbol{s}) + \boldsymbol{x}(\boldsymbol{s})\hat{\boldsymbol{e}}_{\boldsymbol{x}} + \boldsymbol{y}(\boldsymbol{s})\hat{\boldsymbol{e}}_{\boldsymbol{y}}$$

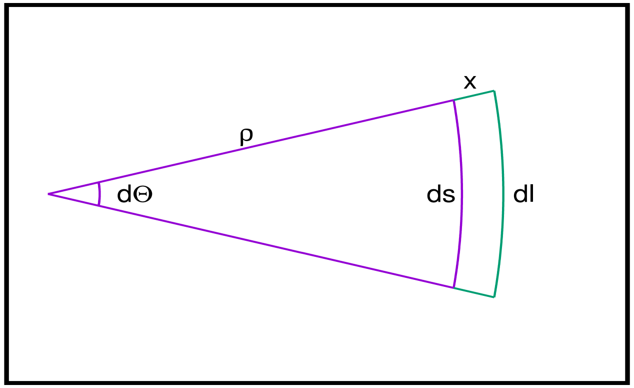
- Use $\boldsymbol{s}$ (distance along the reference trajectory) instead of time

$$\frac{d}{dt} = \frac{d\ell}{dt}\frac{d}{d\ell} = \frac{d\ell}{dt}\frac{ds}{d\ell}\frac{d}{ds}$$

with, for small $\boldsymbol{x'}$ and $\boldsymbol{y'}$,

$$\frac{dl}{dt} \simeq v_s \qquad \text{and} \qquad \frac{d\ell}{ds} \simeq 1 + x/\rho$$

- Linearize fields in the local frame $\{\hat{\boldsymbol{\tau}}, \hat{\boldsymbol{e}}_x, \hat{\boldsymbol{e}}_y\}$



Linearized equations of motion for the generic particle in the local frame attached to the reference trajectory write

$$x'' + \left(\frac{1}{\rho^2} + K\right)x + Ny + 2Hy' = \frac{1}{\rho}\frac{\Delta p}{p} + \frac{e}{p}\Delta B_y$$

$$y'' - Ky + Nx - 2Hx' = \frac{e}{p}\Delta B_x \qquad \leftarrow \quad \text{dipolar errors}$$

with

$$\frac{1}{\rho} = \frac{e}{p}B_y \qquad \leftarrow \quad \text{large } \rho \text{ for large } p$$

$$K(s) \equiv \frac{e}{p}\left(\frac{\partial B_y}{\partial x}\right)_{x=y=0} \quad \text{and} \quad N \equiv \frac{1}{2}\frac{e}{p}\left(\frac{\partial B_x}{\partial x} - \frac{\partial B_y}{\partial y}\right)_{x=y=0}$$

$\uparrow$ normal quad $\qquad\qquad$ $\uparrow$ skew quad

$$H \equiv \frac{1}{2}\frac{e}{p}B_\tau \qquad \leftarrow \quad \text{solenoid}$$

- Solenoids are usually present in the experiment detector, their effect is small, often compensated by anti-solenoids.
  - They may be also machine components for rotating the particle spin direction.
- In general skew quadrupoles are introduced for correction purposes and treated in *perturbation* theory.

Setting $\boldsymbol{N}=\boldsymbol{H}=0$ the two equations are *uncoupled* and, for $\boldsymbol{\Delta p}=0$ and in absence of dipolar errors, may be re-written in a symmetric form (Hill's equation)

$$z'' + K_z z = 0 \qquad \text{with } z = x, y$$

where

quad strength

$$K_x \equiv \left(\frac{1}{\rho^2} + K\right) \qquad \text{and} \qquad K_y \equiv -K \qquad K(s) \equiv \frac{e}{p}\left(\frac{\partial B_y}{\partial x}\right)_{x=y=0}$$

nb: In the following the subscript will be omitted when not needed.

The equations of motion can be derived from the Hamiltonian

$$H = \frac{1}{2}\left[p_x^2 + p_y^2 + K_x x^2 + K_y y^2\right]$$

Twiss functions

Assuming it exists, we use the periodic solution, $\beta(s)$, of

$$\frac{1}{2}\beta\beta'' - \frac{1}{4}\beta'^2 + \beta^2 K = 1$$

and define the new variables η and ϕ

large contribution from small β

$$\eta \equiv \frac{z}{\sqrt{\beta}} \qquad \text{and} \qquad \phi(s) \equiv \frac{1}{Q}\int^s \frac{d\bar{s}}{\beta}$$

The parameter Q is chosen so that

$$\phi(C) = \frac{1}{Q}\oint \frac{ds}{\beta} = 2\pi \ \rightarrow \ Q = \frac{1}{2\pi}\oint \frac{ds}{\beta}$$

We express the derivatives wrt s in terms of the new parameter ϕ

$$\frac{d}{ds} = \frac{1}{Q\beta_0}\frac{d}{d\phi}$$

$$\frac{d^2}{ds^2} = \frac{1}{(Q\beta_0)^2}\frac{d^2}{d\phi^2} + \frac{d}{ds}\left(\frac{1}{Q\beta_0}\right)\frac{d}{d\phi} = \frac{1}{(Q\beta_0)^2}\frac{d^2}{d\phi^2} - \frac{1}{Q^2\beta_0^3}\frac{d\beta_0}{d\phi}\frac{d}{d\phi}$$

The Hill's equation transforms in the equation of a harmonic oscillator

$$\frac{d^2\eta}{d\phi^2} + Q^2\eta = 0$$

The general solution may be written as

$$\eta(\phi) = A\sin(Q\phi + \varphi)$$

The *betatron tune* Q is the number of free oscillations per turn and Q/T_{rev} is the betatron frequency. Back to x and y coordinates:

$$z = A\sqrt{\beta_z(s)}\sin(Q\phi + \varphi)$$

$$z' = \frac{A}{\sqrt{\beta}}[\cos(Q\phi + \varphi) - \alpha\sin(Q\phi + \varphi)]$$

with

$$\alpha \equiv -\frac{1}{2}\frac{d\beta}{ds}$$

- β and ϕ (*Twiss functions*) are properties of the ring optics;
- A and φ are properties of the particle, depending upon the starting conditions.

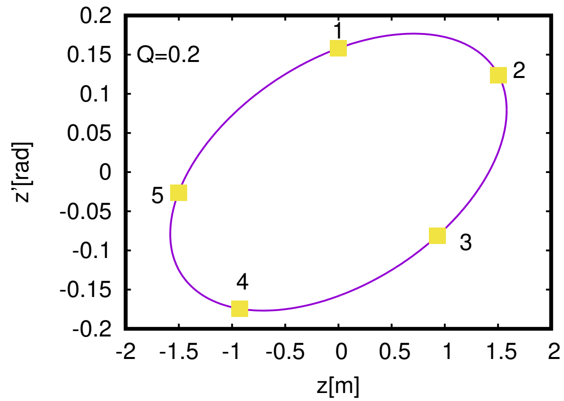
A^2, the *particle emittance*, is a constant of motion.

It is easy to verify that

$$\gamma z^2 + 2\alpha z z' + \beta z'^2 = A^2$$

with $\gamma \equiv (1+\alpha^2)/\beta$.

This is the equation of an *ellipse* in the zz' plane with area πA^2. The shape depends only upon $\beta(s)$ and its derivative, it is the same for all particles.



Beam statistical emittance:

using the expressions for z and z'

$$\epsilon_{rms} \equiv \sqrt{< z^2 >< z'^2 > - < zz' >^2} \overset{\downarrow}{=} \frac{1}{2} < A^2 >$$

In presence of synchrotron radiation, the balance between quantum excitation (from photon emission) and damping (from replenishing the lost energy through RF longitudinal electric fields) results in an *equilibrium* beam emittance.

$$\epsilon_x = \frac{C_q \gamma_{rel}^2}{J_x} \oint ds \frac{1}{|\rho|^3} [\gamma_x D_x^2 + 2\alpha_x D_x D_x' + \beta_x D_x'] \Big/ \oint ds \frac{1}{\rho^2}$$

$$\epsilon_y = \frac{C_q}{J_y} \oint ds \frac{\beta_y}{|\rho|^3} / \oint ds \frac{1}{\rho^2} \simeq C_q \qquad \text{(for a horizontally planar ring)}$$

where

$$C_q = \frac{55}{32\sqrt{3}} \frac{\hbar}{m_0 c} \simeq 3.84 \times 10^{-13} \text{ m}$$

and, usually, $J_x \simeq 1$ and $J_y = 1$. Therefore in general for $e^{\pm}$ it is $\epsilon_y \ll \epsilon_x$.

For p there is only damping from accelerating cavities: emittance *decreases* with energy.

Normalized emittance: $\epsilon_N \equiv \gamma_{rel} \beta_{rel} \epsilon$.

The closed orbit

In presence of dipolar error fields $\frac{e}{p}\Delta B(s) \equiv F(s)$ it is

$$\frac{d^2\eta}{d\phi^2} + Q^2\eta = Q^2\beta^{3/2}F(s(\phi))$$

which is a *forced oscillator*.

The general solution is the sum of the general solution of the homogeneous part and a particular solution of the inhomogeneous equation.

One can verify that

$$\eta(\phi)_{co} = \frac{Q}{2\sin\pi Q}\int_{\phi}^{\phi+2\pi} d\bar{\phi} f(\bar{\phi}) \cos\left[Q(\pi + \phi - \bar{\phi})\right] =$$

$$\frac{Q}{2\sin\pi Q}\int_{0}^{2\pi} d\bar{\phi} f(\bar{\phi}) \cos\left[Q(\pi - |\phi - \bar{\phi}|)\right]$$

with $f(\phi) \equiv \beta^{3/2}F(s(\phi))$, is the *periodic* particular solution.

The *closed orbit* is obtained by multiplying by $\sqrt{\beta}$.

The actual motion of any particle is then described by

$$z = \sqrt{\beta}\eta + z_{co} = A\sqrt{\beta}\sin\left(Q\phi + \varphi\right) + z_{co}$$

with (chopping the integral in the sum of discrete distortions)

$$z_{co}(s) = \frac{1}{2\sin\pi Q}\sqrt{\beta(s)}\sum_j \sqrt{\beta_j}\Theta_j \cos\left[Q(\pi - |\phi(s) - \bar{\phi}_j|)\right]$$

Kick $\rightarrow$ $$\Theta_j \equiv F\Delta s_j = \frac{e}{p}\Delta B_j \Delta s_j$$

- Q can't be an integer number!
- In the linear approximation a closed orbit always exists, it may be out of the vacuum chamber though...
- The effect of errors at large β-value location is particularly strong.
- For correcting the orbit small dipole correctors are inserted in the lattice. Their effect on the closed orbit is of course described by the same expression.

Although today sophisticated methods allow to align the machine components with a precision in the order of some hundred microns, magnet positioning precision is *finite*.

The expectation value of the rms closed orbit can be estimated under the assumption that the number of magnets is large enough

$$< z_{rms} >= \frac{1}{2\sqrt{2}|\sin \pi Q|}\sqrt{< \beta >}\sqrt{\Sigma_i \ \beta_i \ \Psi_i^2}$$

where Ψ_i depends on the kind of error considered. For quadrupole transverse misalignments it is $\Psi = (k\ell)_i \delta z^Q_{rms}$ and

$$< z_{rms} >= \boxed{\frac{1}{2\sqrt{2}|\sin \pi Q|}\sqrt{< \beta >}\sqrt{\sum_{i=1}^{NQ} \ \beta_i (k\ell)_i^2}} \, \delta z^Q_{rms}$$

$\nwarrow$ amplification factor

There are some more "discoveries" related to change of variables from (z, s) to (η, ϕ). For a closed ring, owing to the periodicity, we can expand $z_{co}/\sqrt{\beta} \equiv \eta_{co}$ and $f(\phi) \equiv \beta^{3/2} F(s(\phi))$ in Fourier series

$$\eta(\phi)_{co} = u_0 + \sum_{p=1}^{+\infty} u_p \cos p\phi + v_p \sin p\phi$$

$$f(\phi) = a_0 + \sum_{p=1}^{+\infty} a_p \cos p\phi + b_p \sin p\phi$$

Inserting back into the equation for η we find the relationship between the expansion coefficients

$$\begin{pmatrix} u_p \\ v_p \end{pmatrix} = \frac{Q^2}{Q^2 - p^2} \begin{pmatrix} a_p \\ b_p \end{pmatrix}$$

Under the assumption that the errors have a white spectrum

- The orbit is most sensitive to the harmonics close to the betatron frequency Q.
- The Fourier expansion of an un-corrected closed orbit will have large values of the harmonics close to Q.

Beam Position Monitors are used for measuring the beam transverse position.
The average over many turns gives the closed orbit. Having used ϕ, we see that what matters is the *phase advance*, $Q\phi$ ($\equiv \mu$), rather than the position s:

- BPMs and correctors should be distributed uniformly around the ring wrt to ϕ.
- As $z = \sqrt{\beta}\eta$ and $A = \Theta\sqrt{\beta}$, locations with large β are preferred.

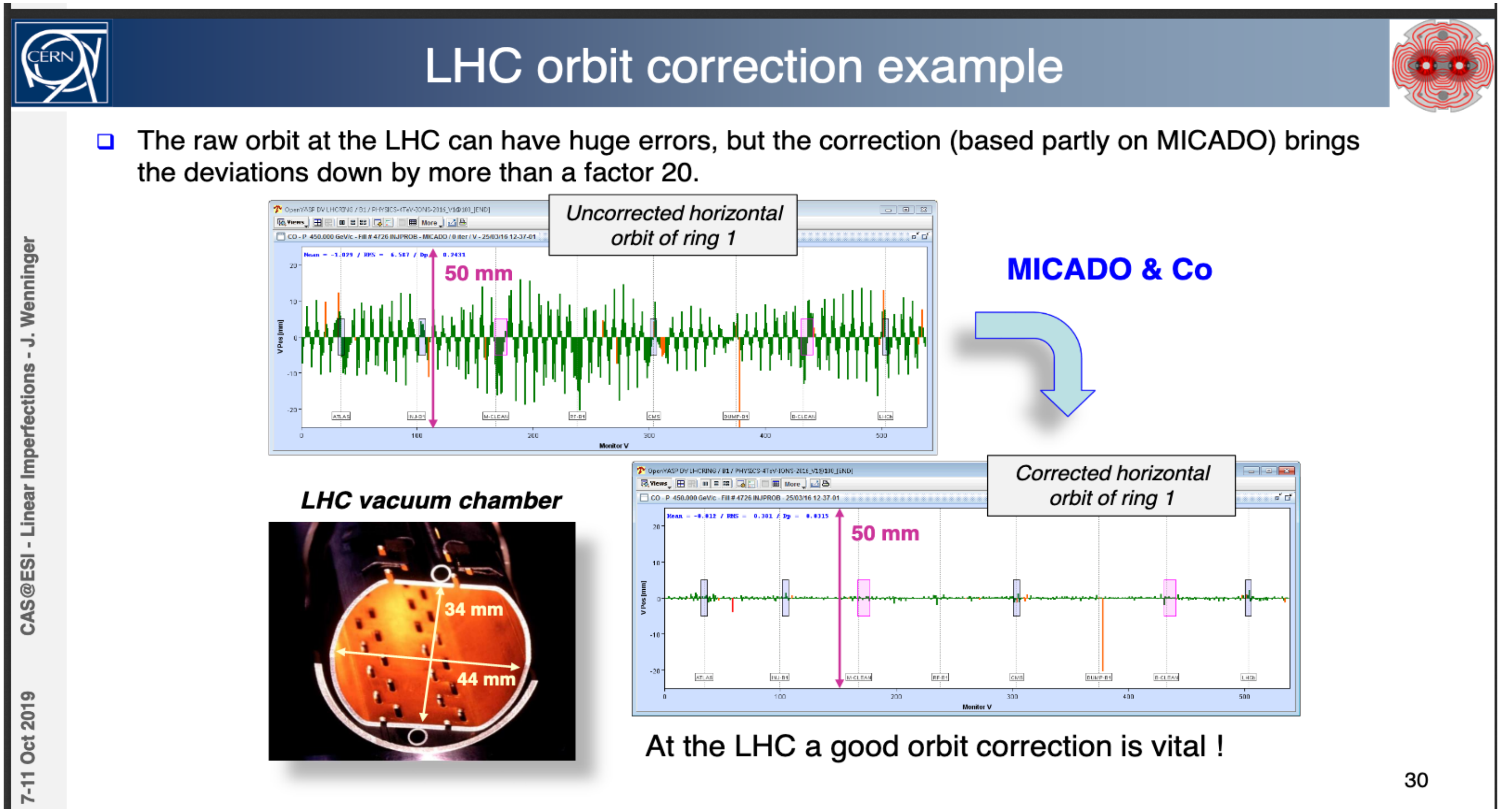
Additional BPMs and correctors must be inserted in special locations as

- injection/extraction region
- for colliders, in the collision region in order to
  - monitor the beam position
  - perform Beam Based Alignment
  - measure the machine optics

It is important to keep the closed orbit as small as possible for

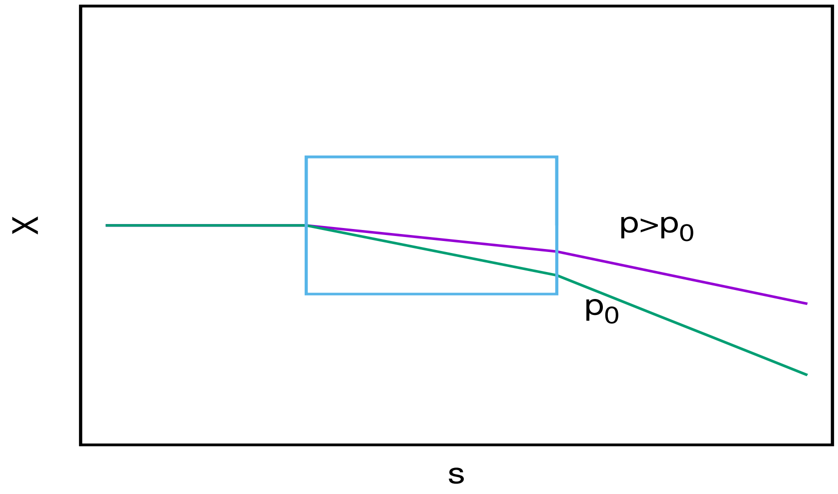
- making good use of the available aperture;
- staying clear of unwanted non-linearities.

An example from the LHC (J. Wenninger, JUAS 2019)



Dispersion

The dispersion, $\boldsymbol{D(s)}$, describes the dependence of the particle trajectory upon its momentum. It originates from the bending magnets.



In a closed ring, the periodic *first order* dispersion is the deviation wrt the design orbit of a particle with an offset $\boldsymbol{\Delta p/p_0}$=1. Therefore

$$\boldsymbol{z = z_\beta + z_{c.o.} + D_z \frac{\Delta p}{p}}$$

The equation of the periodic dispersion has the same form as the equation of the closed orbit, the in-homogeneity being $1/\rho$

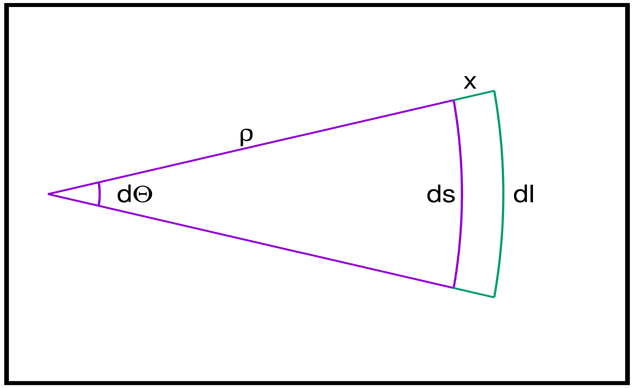
$$D_z'' + K_z D_z = \frac{1}{\rho}$$

The solution has the same form of the closed orbit:

$$D(s) = \frac{\sqrt{\beta(s)}}{2\sin\pi Q}\int_s^{s+L} d\tau \frac{\sqrt{\beta(\tau)}}{\rho}\cos\left[Q(\pi + \phi(s) - \phi(\tau))\right]$$

Particles with $\Delta p \neq 0$ have a different path length:

$$L = L_0 + \Delta L = L_0 + \oint ds \frac{x}{\rho} = L_0 + \frac{\Delta p}{p}\oint ds \frac{D}{\rho}$$



Momentum compaction factor

$$\alpha_p \equiv \frac{dL}{L} / \frac{dp}{p}$$

β-beating

We recall the equation for the β function used to disentangle the motion of the single particles from the machine characteristics:

$$\frac{1}{2}\beta_z\beta_z'' - \frac{1}{4}\beta_z'^2 + \beta_z^2 K_z = 1 \qquad (z = x, y)$$

with

$$K_x \equiv \left(\frac{1}{\rho^2} + K\right) \qquad \text{and} \qquad K_y \equiv -K$$

and

$$K(s) \equiv \frac{e}{p}\left(\frac{\partial B_y}{\partial x}\right)_{x=y=0}$$

Errors in the quadrupole strength lead to a β perturbation.

A direct way [a] for finding the equation describing the perturbation in first approximation consists in

- writing $\boldsymbol{K} = \boldsymbol{K_0} + \boldsymbol{\Delta K}$ and $\boldsymbol{\beta} = \boldsymbol{\beta_0} + \boldsymbol{\Delta\beta}$;
- inserting those expressions in the $\boldsymbol{\beta}$ function equation;
- recognizing that $\boldsymbol{\beta_0}$ is the unperturbed solution to the unperturbed equation;
- keeping only the linear terms in $\boldsymbol{\Delta K}$ and $\boldsymbol{\Delta\beta}$;
- using $\boldsymbol{\beta_0\beta_0'' + 2\beta_0^2 K_0 = 2(1 + \beta_0'^2/4)}$ (from the unperturbed equation);
- writing the derivatives wrt $\boldsymbol{s}$ in terms of $\boldsymbol{\phi}$ (as shown when deriving the equation for $\boldsymbol{\eta = z/\sqrt{\beta}}$).

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[a] In the literature it is in general obtained by introducing a thin lens perturbation in the one turn transport matrix.

The β-beat equation, which looks awful when s is used, takes a simple and enlightening form when ϕ is used

$$\frac{d^2}{d\phi^2}\left(\frac{\Delta\beta}{\beta_0}\right) + 4Q^2\left(\frac{\Delta\beta}{\beta_0}\right) = -2Q^2\beta_0^2\Delta K(\phi)$$

It has the same form as the closed orbit equation for $\eta \equiv z/\sqrt{\beta}$ with

$$Q \to 2Q$$

$$Q^2\beta^{3/2}F(\phi) \to -\frac{1}{2}(2Q)^2\beta_0^2\Delta K(\phi)$$

and we can use the results found for η

$$\frac{\Delta\beta}{\beta_0}(s) = -\frac{1}{2\sin(2\pi Q)}\int_0^L d\bar{s}\,\beta\Delta K\cos 2Q[\pi - |\phi(s) - \phi(\bar{s})|]$$

For one integrated gradient error, $\Delta K\ell$, at $s = s_k$

$$\frac{\Delta\beta}{\beta_0}(s) = -\frac{\beta_k\Delta K\ell}{2\sin(2\pi Q)}\cos 2Q[\pi - |\phi(s) - \phi_k|]$$

$\Delta\beta/\beta_0(s)$ (beta-beat)

- oscillates with *twice* the betatron frequency and thus is sensitive to gradient error harmonics near to $2Q$ $\rightarrow$ true β-beat is reach in harmonics close to $2Q$;
- is large when Q approaches a *half integer*.

Tune in presence of quadrupole error can be found from the definition

$$Q = \frac{1}{2\pi}\oint \frac{ds}{\beta} = \frac{1}{2\pi}\oint \frac{ds}{\beta_0 + \Delta\beta} \simeq Q_0 - \frac{1}{2\pi}\oint \frac{ds}{\beta_0}\frac{\Delta\beta}{\beta_0} = Q_0 - \frac{1}{2\pi}\oint d\mu \frac{\Delta\beta}{\beta_0}$$

with $\mu \equiv Q_0\phi$.

For a localized gradient error

$$\oint d\mu \frac{\Delta\beta}{\beta_0} = \frac{\beta_k \Delta K \ell}{2\sin(2\pi Q)}\oint d\mu \cos[2Q\pi - 2|\mu - \mu_k|] = \frac{1}{2}\beta_k \Delta K \ell$$

$$\Delta Q = -\frac{1}{4\pi}\beta_k \Delta K \ell \qquad \text{(tune shift)}$$

The effect of more gradient errors add linearly.

Chromaticity

We can make use of this result to easily find the equation of the (linear) chromaticity.

Particles with a momentum difference wrt the nominal one, experience different forces:

$$\frac{1}{\rho} = \frac{e}{p} B_y \qquad \text{and} \qquad K = \frac{e}{p} \Big(\frac{\partial B_y}{\partial x} \Big)_{x=y=0}$$

Using

$$\frac{1}{p} = \frac{1}{p_0 + \Delta p} = \frac{1}{p_0} \Big[1 - \frac{\Delta p}{p_0} + \Big(\frac{\Delta p}{p_0} \Big)^2 + \dots \Big]$$

and keeping the linear term only, it is

quadrupole error

$$K = K_0 + \Delta K \simeq K_0 - \frac{\Delta p}{p_0} K_0$$

Chromatic (linear) β-beating

$$\frac{1}{\Delta p / p_0} \frac{\Delta \beta}{\beta} = \frac{1}{2 \sin(2\pi Q)} \int_0^L d\bar{s}\, \beta K_0 \cos 2Q [\pi - |\phi(s) - \phi(\bar{s})|]$$

The largest contribution comes from strong quadrupoles at large β.

The tune shift due to a single quadrupole for a particle with momentum $p_0 + \Delta p$ is

$$(\Delta Q)_k = \frac{1}{4\pi}\beta_k \Delta K \ell = -\frac{1}{4\pi}\beta_k \frac{\Delta p}{p_0} K_0 \ell$$

By integrating over the machine length we get the total tune change

$$\Delta Q = -\frac{1}{4\pi}\frac{\Delta p}{p_0}\oint ds \beta K_0$$

The (natural) linear chromaticity is

$$\xi \equiv \frac{\Delta Q}{\Delta p / p_0} = -\frac{1}{4\pi}\oint ds \beta K_0$$

The chromaticity is larger in large rings and the largest contributions come from

- quadrupoles at large β location;
- strong quadrupoles.

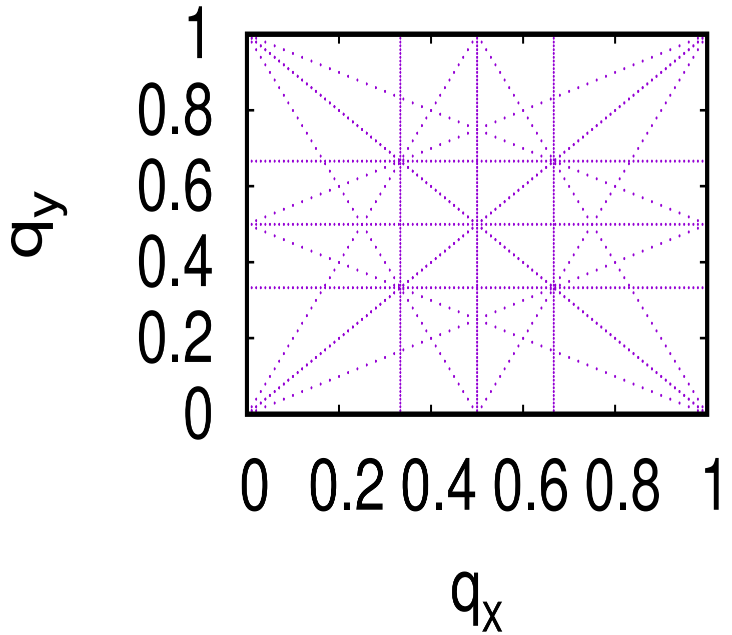
- The negative natural chromaticity drives the Head-Tail instability.
- Large chromaticity means that there may be no stable optics for off-momentum particles.
- Chromaticity leads to a spread of the particle tunes. Particles with tune close to a resonance [a]

$$n_x Q_x + n_y Q_y = p$$

(n_z integer) may be lost.

[a] depending whether there are field driving the resonance.

Working point diagram with resonances up to 3d order.



We need one more building block: a quadrupole which strength depends linearly on momentum!

Sextupole magnets placed in locations where $D_x \neq 0$ are exactly that.

quadrupole term

$$B_x = Sxy = S(D_x \frac{\Delta p}{p_0} + x_\beta) y_\beta = \boxed{S D_x \frac{\Delta p}{p_0}\, y_\beta} + S x_\beta y_\beta$$
$$B_y = \frac{1}{2} S(x^2 - y^2) = \frac{1}{2} S(D_x \frac{\Delta p}{p_0})^2 + \boxed{S D_x \frac{\Delta p}{p_0} x_\beta} + \frac{1}{2} S(x_\beta^2 - y_\beta^2)$$

The simplest correction scheme consists in placing sextupoles in the arcs, where $D_x \neq 0$,

$$\Delta Q_x = \frac{1}{4\pi} \sum_{i=1}^{NS} \beta_{x,i} D_{x,i} S_i \ell_i$$
$$\Delta Q_y = \frac{1}{4\pi} \sum_{i=1}^{NS} \beta_{y,i} D_{x,i} S_i \ell_i$$

Arranging the sextupoles into two families, the values of their strength, S_F and S_D, for a given ΔQ_x and ΔQ_y are obtained by inverting a system of two equations

$$\begin{pmatrix} \Delta Q_x \\ \Delta Q_y \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix} \begin{pmatrix} S_F \\ S_D \end{pmatrix}$$

Sextupoles introduce also unwanted effects:

- The dipole-like term $(D_x \frac{\Delta p}{p_0})^2$ (*chromatic aberration*) introduces 2d order dispersion.
- The terms $\frac{1}{2} S(x_\beta^2 - y_\beta^2)$ and $S x_\beta y_\beta$ (*geometric aberration*). They drive 3d order resonances, unless attention is paid to the phase advance between sextupoles of the same family so that an overall cancelation occurs.
  - In a non-linear machine not all amplitudes are stable: the maximum amplitude that can circulate is no more determined by the beam pipe (*physical aperture*). Computer codes are used to perform tracking and determine the *dynamic aperture*.

For minimizing their strengths (and troubles) it is convenient to place the sextupoles at location where D_x is large and $\beta_z^2 \gg \beta_x \beta_y$ so that the corrections are (almost) orthogonal.

In practice we may consider the ring as made of piece-wise constant fields.

Example of Hill's equation for a pure quadrupole.

$$x'' + Kx = 0$$
$$y'' - Ky = 0$$

where K=*const*. The equations are easily solved. For x

$$x = A\cos\sqrt{K}s + B\sin\sqrt{K}s \qquad \text{for} \quad K > 0$$
$$x = A\cosh\sqrt{|K|}s + B\sinh\sqrt{|K|}s \qquad \text{for} \quad K < 0$$

with

$$A = x(0)$$
$$B = x'(0)/\sqrt{|K|}$$

The quadrupole is *horizontally focusing* if $K > 0$, *horizontally defocusing* if $K < 0$.

The effect of the same quadrupole is *inverted* in the vertical direction.

The solution may be expressed in matrix form.

For instance for a horizontally focusing quadrupole it is for x

$$\begin{pmatrix} x \\ x' \end{pmatrix} = \begin{pmatrix} \cos\sqrt{K}s & \frac{\sin\sqrt{K}s}{\sqrt{K}} \\ -\sqrt{K}\sin\sqrt{K}s & \cos\sqrt{K}s \end{pmatrix} \begin{pmatrix} x_0 \\ x'_0 \end{pmatrix}$$

We notice that the matrix determinant is 1.

The one turn transport matrix, $M(s+L, s)$, is obtained multiplying the single machine elements matrices.

The motion is stable only if the eigenvalues of the one turn matrix are complex-conjugates modulus 1. If the condition is satisfied, it is possible to define the Twiss functions.

Relationship between matrix elements and Twiss functions:

$$M(s, s_0) = \begin{pmatrix} \frac{\sqrt{\beta}}{\sqrt{\beta_0}}(\cos\Delta\mu + \alpha_0 \sin\Delta\mu) & \sqrt{\beta_0\beta}\sin\Delta\mu \\ -\frac{1}{\sqrt{\beta_0\beta}}[(\alpha-\alpha_0)\cos\Delta\mu + (1+\alpha_0\alpha)\sin\Delta\mu] & \frac{\sqrt{\beta_0}}{\sqrt{\beta}}(\cos\Delta\mu - \alpha\sin\Delta\mu) \end{pmatrix}$$

with $\Delta\mu$=$\mu(s)$-$\mu(s_0)$. The matrix determinant is 1. This is a consequence of the matrix being *symplectic*. Any Hamiltonian system is symplectic.

Starting for simplicity from a symmetry point where α_0=0, the one-turn matrix is

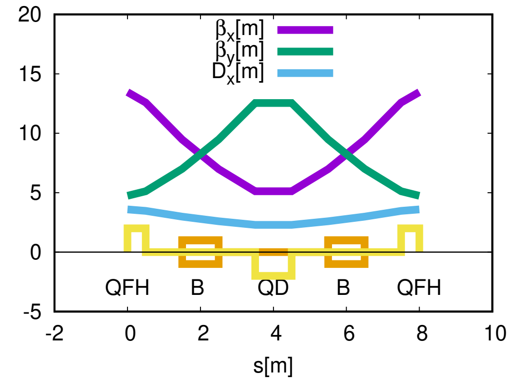
$$M(s+L,s) = \begin{pmatrix} \cos 2\pi Q & \beta \sin 2\pi Q \\ -\frac{1}{\beta}\sin 2\pi Q & \cos 2\pi Q \end{pmatrix}$$

The eigenvalues are indeed

$$\lambda^{\pm} = e^{\pm i 2\pi Q}$$

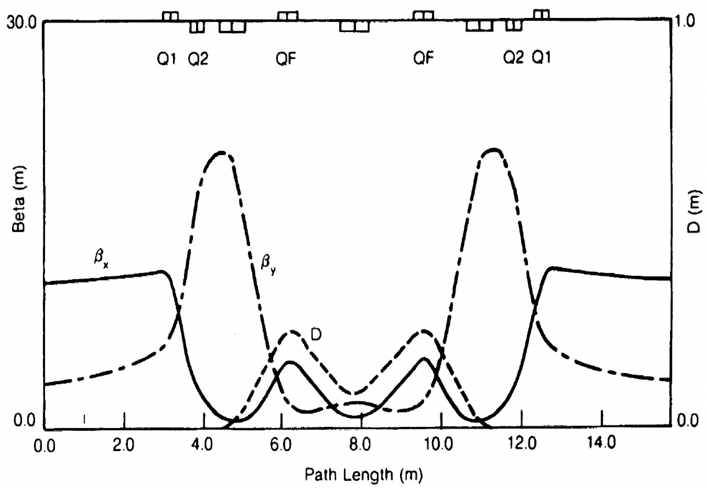
The matrix formalism is useful for practical calculations.

We may try building a closed ring! Quadrupole focusing in one plane are defocusing in the other one: a sequence of horizontally (QF) and vertically focusing (QD) quadrupoles are needed for overall focusing. The quadrupole layout depends on the purpose.



- FODO cells are very common
  - inserting dipoles between the quadrupoles, FODO cells are convenient for building the ring arcs.
- Light sources aiming to very small e^- beam sizes use more sophisticated cells: Double Bend Achromat, Triple Bend Achromat...

A. JACKSON



Insertions

Insertions are needed for

- Injection and possibly extraction.
- Accommodating RF cavities.
- Collimation.
- Connecting dispersive sections to dispersion-free ones (*dispersion suppressor*).
- Creating enough free space for experiments in colliders.

  Crucial parameters for colliders:

  – Energy in the Center of Mass.

    For two ultra-relativistic particles colliding "head-on": $E_1' + E_2' = 2\sqrt{E_1 E_2}$

  – Luminosity, $\mathcal{L}$.

    Rate of events with a given cross section, σ, is

$$R = \mathcal{L} \times \sigma$$

Low β-insertions

Luminosity (head-on collisions):

# of particles in beam 1 # of particles in beam 2

$$\boldsymbol{\mathcal{L}} = \frac{\boldsymbol{N_1 N_2}}{\boldsymbol{A}} \boldsymbol{f_{coll}} \quad \leftarrow \text{ collision frequency}$$

beam transverse area

The effective transverse area for Gaussian distributions, and assuming the two beams have the same sizes, is

$$\boldsymbol{A = 4\pi\sigma_x\sigma_y}$$

with $\boldsymbol{\sigma_z = \sqrt{\beta_z \epsilon_z}}$.

If each beam contains n_b bunches it is $f_{coll} = n_b f_{rev}$. Luminosity may be rewritten as

\# particles/bunch

$$\mathcal{L} = \frac{1}{4\pi} \frac{N_1 \, N_2}{\sigma_x^* \sigma_y^*} n_b f_{rev}$$

For maximizing the luminosity we can

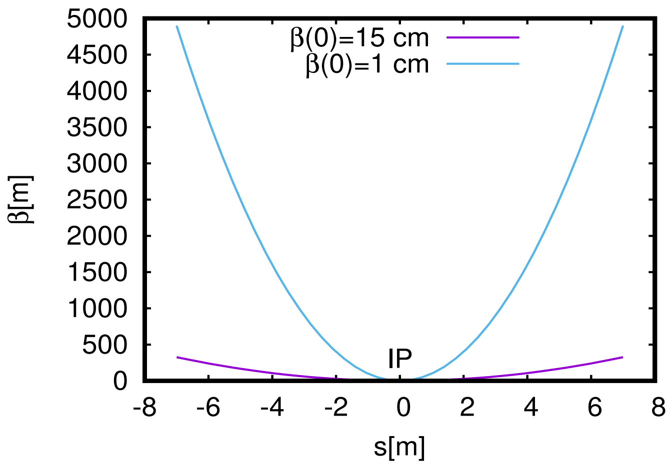
- Push the bunch population to an instability limit.
- Minimize the emittance.
  - For hadrons it means preserving the emittance (injection matching, avoid resonances, etc) and beam cooling.
  - For leptons, when emittance is dictated by synchrotron radiation: optics design.
- Minimize β_z at the Interaction Point, the so-called β_z^*.

There are limitations to how small β^* can be made.

β function in a drift space:

$$\beta(s) = \beta_0 - 2\alpha_0 s + \gamma_0 s^2 \qquad \alpha \equiv -\tfrac{1}{2}\beta'$$

$$\gamma \equiv (1+\alpha^2)/\beta$$



For small β_0 and α_0=0

$$\beta(s) \approx \frac{s^2}{\beta_0}$$

Quadrupole focal length

$$f \approx \frac{1}{K\ell_q} = L_{free} \qquad \text{with} \quad 2L_{free} = \text{ machine magnet-free space}$$

$$\hat\beta \approx \frac{L_{free}^2}{\beta^*} = \frac{1}{(K\ell_q)^2}\frac{1}{\beta^*}$$

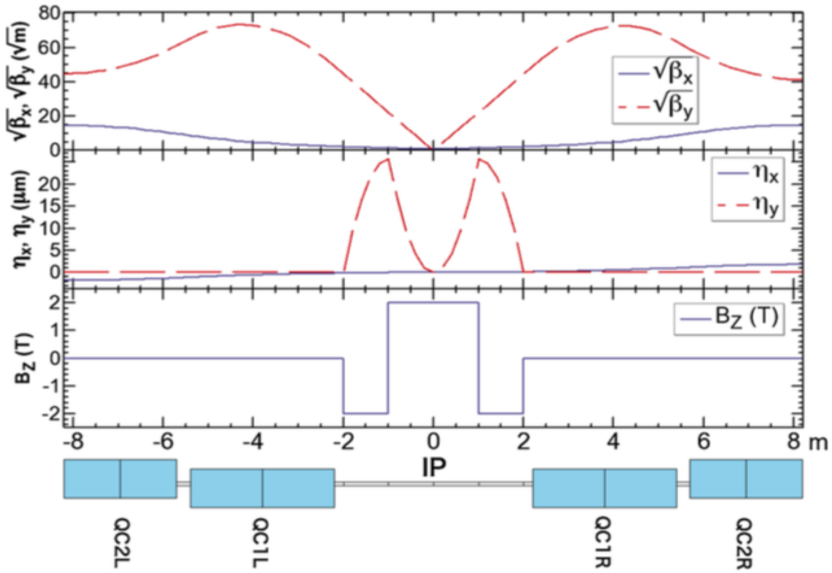
Contribution to linear natural chromaticity:

$$\Delta Q \propto \hat\beta K\ell_q = \frac{\hat\beta}{L_{free}} = \frac{L_{free}}{\beta^*}$$

- L_{free} should be as small as possible.
- The IR quadrupoles
  - will be strong;
  - their aperture must accommodate large beam transverse size;
  - must be well aligned (closed orbit!);
  - have good field quality (small multipoles);
- The bunch length must be not larger than β^*, as β strongly increases moving away from the IP reducing luminosity (*Hourglass effect*).

Example. Future Circular Collider e^+e^- ring uses ultra-flat beam.

K. Oide et al, (PRAB 19, 111005 (2016)



Optics	β_x	β_y	ϵ_x	ϵ_y
	[mm]	[mm]	[nm]	[pm]
45 GeV	150	0.8	0.3	1.0
80 GeV	200	1.0	0.8	1.7

Optics		vertical orbit mplification factor
45 GeV	all quads	665
(Jan.2018)	w/o IRs quads	124

- The few IRs quadrupoles have a huge effect on the closed orbit.
- It is crucial to have near-by BPMs and correctors to compensate their effects *locally*.

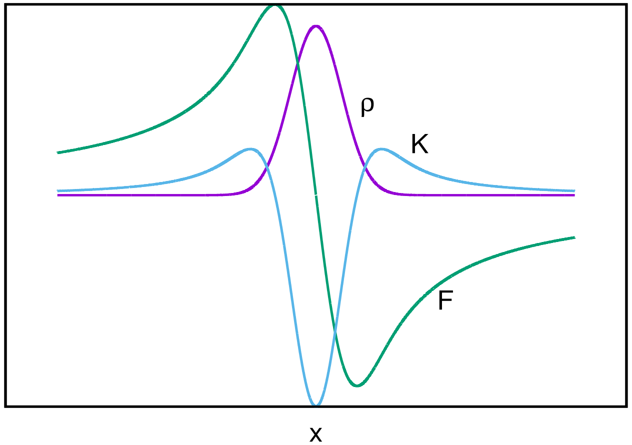
We have seen that strong quadrupoles at large β function values are the main contributors to chromaticity.

Example. Future Circular Collider e^+e^- ring.

Optics		ξ_x^{nat}	ξ_y^{nat}
45 GeV	all sexts off	-361	-1540
	IR setxs off	+3.5	-1230
80 GeV	all sexts off	-359	-1331
	IR setxs off	+3	-1017

$$\xi^{nat} = -\frac{1}{4\pi}\oint ds\, \beta(s)K(s)$$

A second limitation comes from beam-beam effects: each beam acts on the counter-rotating one as a (non-linear) lens.



The *incoherent* beam-beam tune shift, χ_z, is the tune change for a particle at the center of the distribution:

$$\chi_z = \frac{r_c N}{2\pi\gamma_{rel}} \frac{\beta_z^*}{\sigma_z^*(\sigma_x^* + \sigma_y^*)}$$

r_c classical radius of the particle

N #particles in the counter-rotating bunch

Imposing $\chi_x = \chi_z \quad \Rightarrow \quad \frac{\beta_y^*}{\beta_x^*} = \frac{\epsilon_y}{\epsilon_x}$

e^+e^--colliders: $\epsilon_y \ll \epsilon_x \quad \Rightarrow \quad \beta_y^* \ll \beta_x^*$

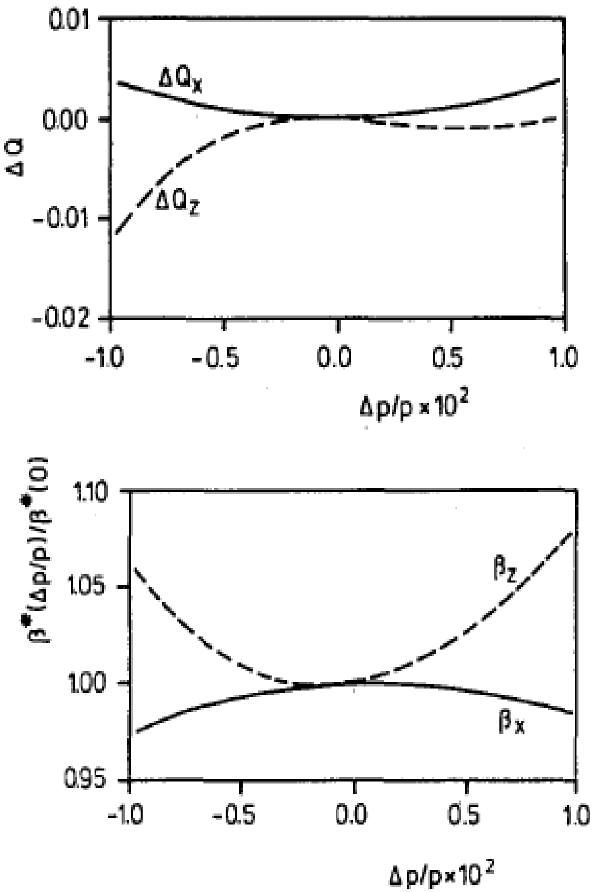
Round beams (non radiating particles): $\beta_x^* = \beta_y^*$

Low-β insertion chromaticity correction

Relative energy spread: $\approx 1‰$@ 27 GeV.

In colliders where the IR chromaticity is not very large, using sextupoles in the arcs is satisfactory.
For instance in (pre-upgrade) HERA-e, 3 families of sextupoles/arc in the 60^0 FODO cells could be used for correcting

- linear chromaticity
- chromatic β-beating at the IPs.

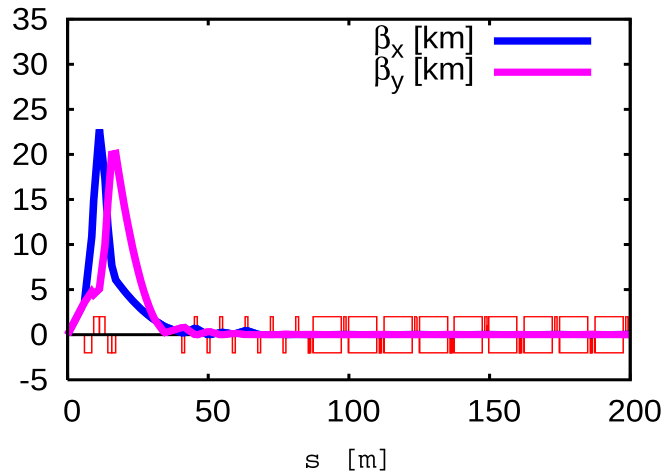


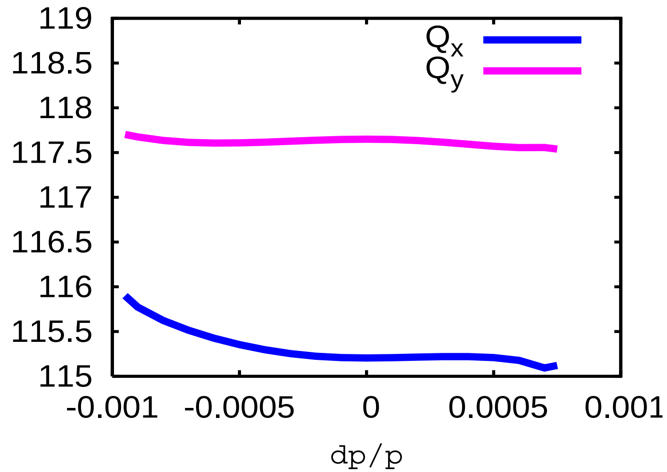
Brinkmann-Willeke, HEACC 1986

This scheme is not adequate for highly chromatic machines.

Example of a 1.5 TeV Muon Collider.

Design Parameters	
E_{beam}	750 GeV
Lenght	2.5 Km
N_b × Num. of muons/bunch	$1 \times 2 \cdot 10^{12}$
$\epsilon^N_{x,y} = \beta_{rel}\gamma_{rel}\epsilon_{x,y}$	25 μm
Δp/p	0.1 %
Average Luminosity	1.25×10^{34} cm^{-2} s^{-1}
β^*_x, β^*_y	1 cm
Number of IPs	2
beam-beam tune shift/IP	0.1
$\lvert\alpha_p\rvert$	$<1 \times 10^{-4}$





$\rightarrow \Delta p/p \simeq \pm 0.08$ %
with a HERAe-like scheme.

For improving the stability range, dipoles have been introduced close to the Final Focus quads allowing local chromaticity correction. Montague chromatic functions $\boldsymbol{W_{x,y}}$:

$$\boldsymbol{W_z \equiv \sqrt{A_z^2 + B_z^2}}$$

$$\boldsymbol{A_z \equiv \frac{\partial \alpha_z^{(0)}}{\partial \delta_p} - \alpha_z^{(0)} B_z} \qquad \boldsymbol{B_z \equiv \frac{1}{\beta_z^{(0)}} \frac{\partial \beta_z}{\partial \delta_p}} \qquad \boldsymbol{(z = x/y)}$$

$\boldsymbol{\Delta p/p}$

$$\boldsymbol{\frac{dA_z}{ds} = 2B_z \frac{d\mu_z^{(0)}}{ds} - \beta_z^{(0)} k} \quad \text{and} \quad \boldsymbol{\frac{dB_z}{ds} = -2A_z \frac{d\mu_z^{(0)}}{ds}}$$

$$\boldsymbol{k} \equiv \begin{cases} \boldsymbol{+(K - D_x S)} & \text{(hor.)} \\ \boldsymbol{-(K - D_x S)} & \text{(vert.)} \end{cases} \qquad \begin{array}{l} \boldsymbol{K} \equiv \text{quad. strength} \\ \boldsymbol{S} \equiv \text{sext. strength} \end{array}$$

- $\boldsymbol{A_z(s)}$ becomes non-zero when going from the IP ($\boldsymbol{A_z}$=$\boldsymbol{B_z}$=0) through the first FF quad.
- $\boldsymbol{B_z(s)}$=0 as long as $\boldsymbol{d\mu_z^{(0)}/ds}$=0.

A sextupole close to the FF quads (large $\boldsymbol{\beta_z \rightarrow d\mu_z^{(0)}/ds}$=0) corrects $\boldsymbol{A_z}$ and keeps $\boldsymbol{B_z}$=0.

Second order chromaticity

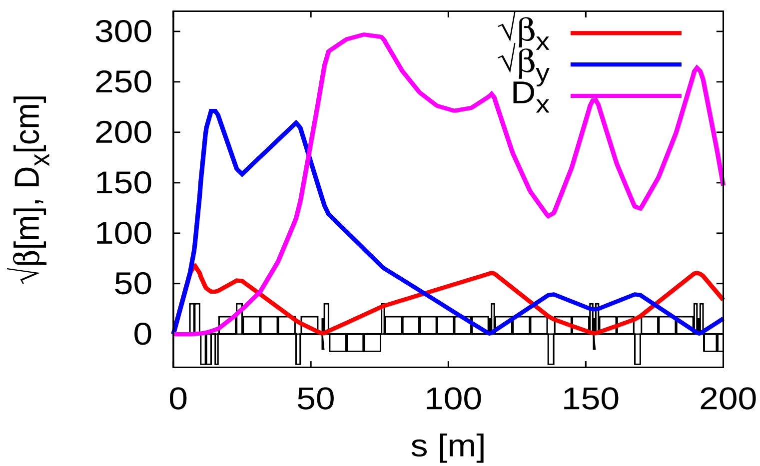
$$\xi_z^{(2)} = \frac{1}{8\pi}\int_0^C ds\left(-kB_z \pm 2S\frac{dD_x^{(0)}}{d\delta_p}\right)\beta_z^{(0)} - \xi_z^{(1)}$$

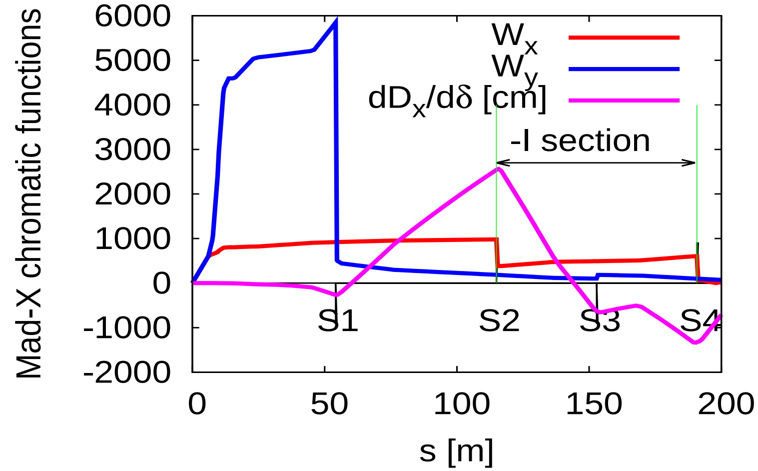
lin. chrom.

$\rightarrow$ chromatic functions $B_{x,y}$ and $dD_x^{(0)}/d\delta_p$ must be both compensated!

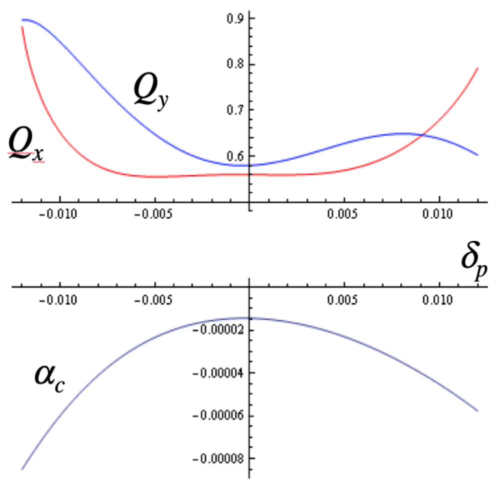
- With $\hat\beta_y \gg \hat\beta_x$ (focusing first in the horizontal plane)
  - W_y is first corrected by a single sextupole at $\Delta\mu_y \approx 0$ from IP and very small β_x (for normal sextupole it ensures that the effect on detuning with amplitude and resonance driving terms are small, a consequence of $H = ax^3 - 3axy^2$).
  - W_x is corrected with one sextupoles at $\Delta\mu_x = m\pi/2$ from IP and $\beta_x \gg \beta_y$;
    - a "twin" sextupole at (pseudo)$-I$ reinforces its FF chromatic β-wave correction, while canceling its aberrations.
- 2d order dispersion may be corrected by sextupoles at a low $\beta_{x,y}$ locations.

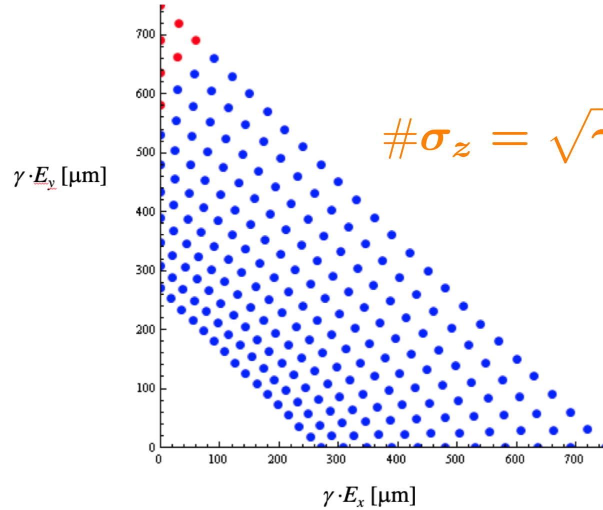
Interaction region with a doublet FF with L_{free}=6 m for E_{beam}=750 GeV.





(Y. Alexahin et al.)





$\#\sigma_z = \sqrt{\gamma E_z / \epsilon^N}$

- Momentum acceptance of ±1.2% exceeds requirement.
- DA (on energy) is ≈ 5 σ ($\epsilon_\perp^N$ 25 μm).

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THANKS!
Enjoy the conference.

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