Beam Dynamics in ElectroMagnetic Field

Very first BASICS

Nicolas PICHOFF

CEA-Saclay
France

Contents

Particle representation

Particle definition

(considered as known)

A **rest mass *m*** and a **charge *q*** $\longrightarrow$ $\boxed{q/m}$

In 3D-dynamics world, the particle is a **dot** at :

$\overrightarrow{\textbf{Position}}$: $\vec{\boldsymbol{r}}$

$\overrightarrow{\textbf{Motion}}$: change of $\vec{\boldsymbol{r}}$ with respect to an **independent variable** (**time**: t)

Motion

Velocity : $\vec{\boldsymbol{v}} = \dfrac{d\vec{r}}{dt}$

$c = 299\,792\,458$ m/s

Momentum : $\vec{\boldsymbol{p}} = \gamma \cdot m \cdot \vec{v}$

Total energy : $\boldsymbol{W}$

$$W^2 = (mc^2)^2 + (\vec{\boldsymbol{p}} \cdot \vec{\boldsymbol{p}})c^2$$

Kinetic energy : $\boldsymbol{K}$

$$= (mc^2 + K)^2$$

Reduced

Velocity : $\vec{\boldsymbol{\beta}} = \dfrac{\vec{v}}{c}$

Momentum : $\overrightarrow{\boldsymbol{\beta\gamma}} = \dfrac{\vec{p}}{mc}$

$$\gamma^2 = \frac{1}{1 - \beta^2} = 1 + (\beta\gamma)^2$$

Energy : $\boldsymbol{\gamma} = \dfrac{W}{mc^2}$

Particle Motion (Lorentz)

(considered as known)

At a given t, a particle is represented by a 6-coordinates vector :

$$\begin{pmatrix}\vec{r}\\ \vec{p}\end{pmatrix} = \begin{pmatrix} x \\ y \\ z \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

The evolution (with independent variable t) of these vectors are given by :

$$\begin{cases} \dfrac{d\vec{p}}{dt} = \vec{F} = q \cdot \left(\vec{E} + \vec{v} \times \vec{B}\right) \\ \dfrac{d\vec{r}}{dt} = \dfrac{\vec{p}}{\gamma m} \end{cases}$$

With : $\gamma = \sqrt{1 + \left(\dfrac{\|\vec{p}c\|}{mc^2}\right)^2}$

$\vec{E}$ and $\vec{B}$ are the **electromagnetic fields** produced by external sources and the particles (position and motion) themselves.

Energy gain

The energy evolution with time t is :

$$\left(W^2 = (mc^2)^2 + (\vec{p}\cdot\vec{p})c^2\right)$$

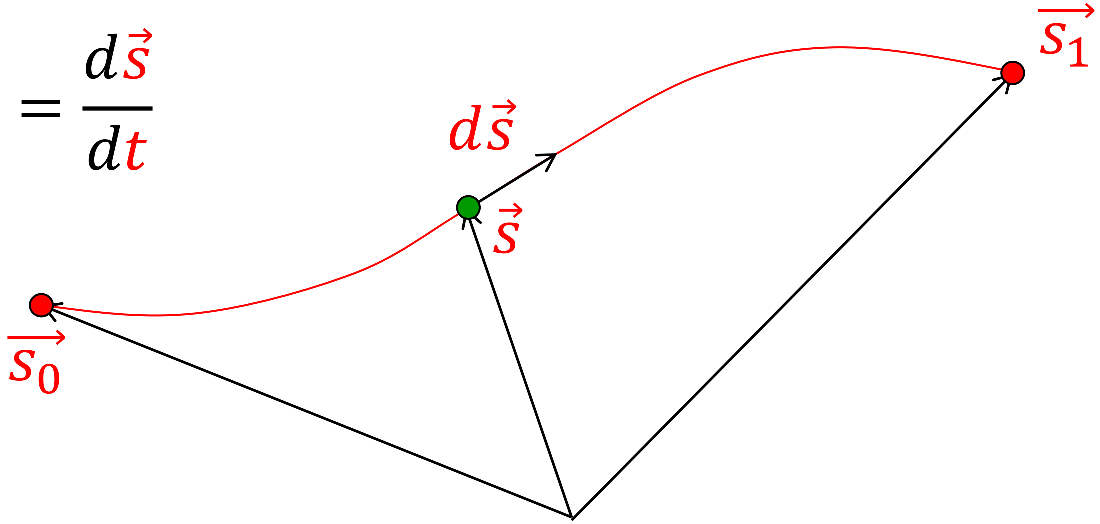
$$\frac{dW}{dt} = \frac{\vec{p}}{W}\cdot\frac{d\vec{p}}{dt}\cdot c^2 = \frac{\cancel{\gamma m}\vec{v}}{\cancel{\gamma m c^2}}\cdot q\cdot\left(\vec{E}+\vec{v}\times\vec{B}\right)\cdot\cancel{c^2}$$

$$= q\cdot\left(\vec{v}\cdot\vec{E} + \underbrace{\vec{v}\cdot\left(\vec{v}\times\vec{B}\right)}_{=0}\right)$$

$$= q\cdot\vec{v}\cdot\vec{E} \quad\Rightarrow$$

Only the electric field gives (kinetic) energy to the beam

$$\vec{v} = \frac{d\vec{s}}{dt}$$



$$\Delta W\left(\overrightarrow{s_0}\to\overrightarrow{s_1}\right) = q\cdot\int_{\overrightarrow{s_0}}^{\overrightarrow{s_1}}\vec{E}\left(\vec{s},t\right)\cdot d\vec{s} \quad\Rightarrow$$

The energy gain is the path integral of the electric field

Particle accelerator

Acceleration/deviation

An accelerator is a **set of technological components** producing ElectroMagnetic fields allowing to accelerate and guide particles.

$$\left(\vec{E};\vec{B}\right)$$

Acceleration is produced (generaly) by time varying electric field

$$\vec{E}\;(\vec{r},t)$$

- RF cavities
- Plasma
- Exception @ low energy : *Electrostatic*

Guiding is produced (generaly) by static magnetic field

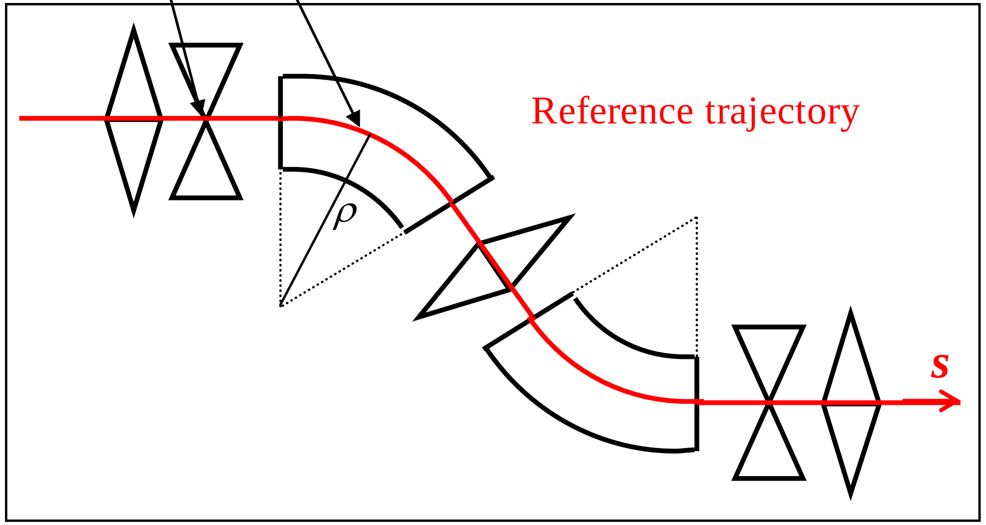
$$\vec{B}(\vec{r})$$

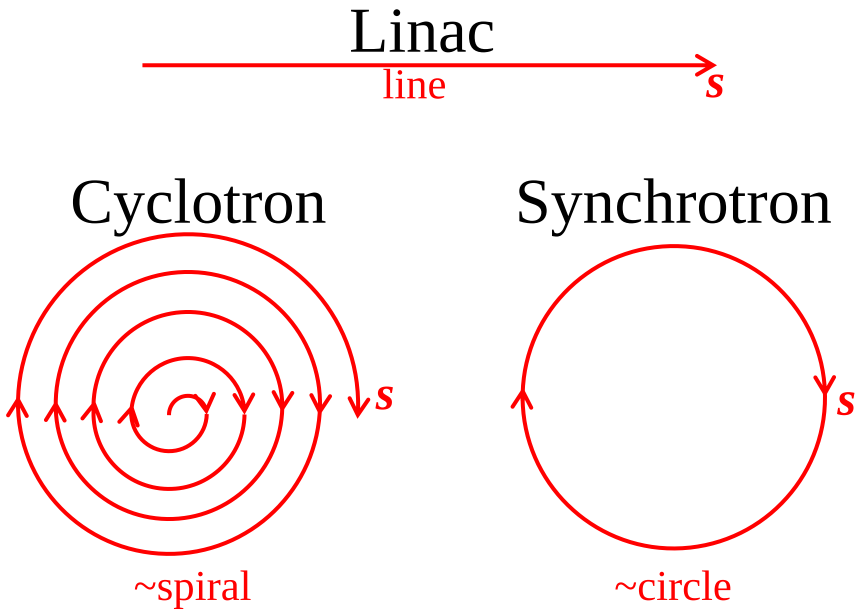
- *Deviation* with *Dipoles*
- *Focalisation* with *Quadrupoles* or *Solenoids*
- *Stabilisation* with higher-order *Multipoles*
- Exception @ low energy : *Electrostatic*

Reference trajectory

An accelerator is designed around a **reference trajectory** (design orbit in circular accelerators), which is:

- A strait line in drift, focusing or accelerating elements,
- An arc of circle in dipole magnets.





The beam dynamics is computed along the reference trajectory as a function of the **curved abscissa : s**, replacing t as the *independent variable*.

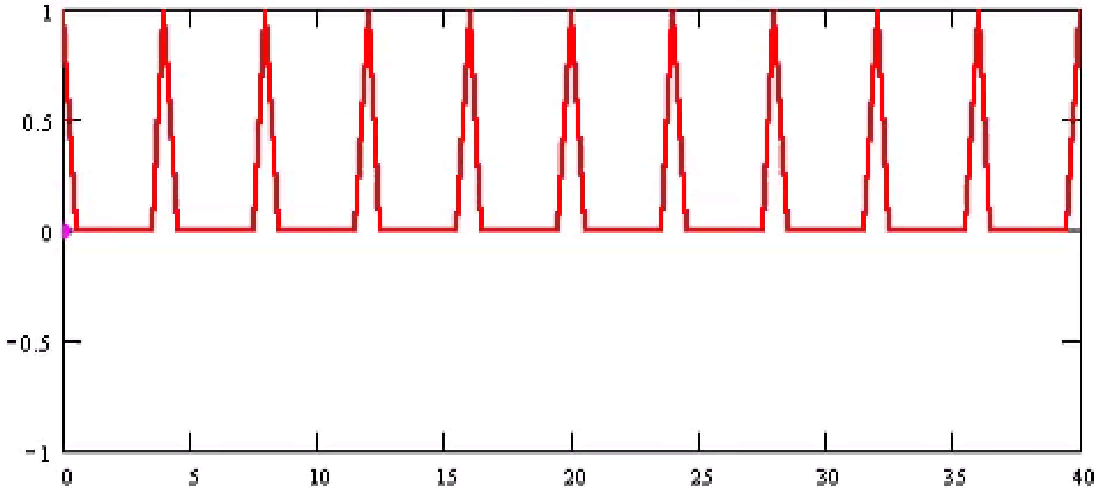
Synchronous particle

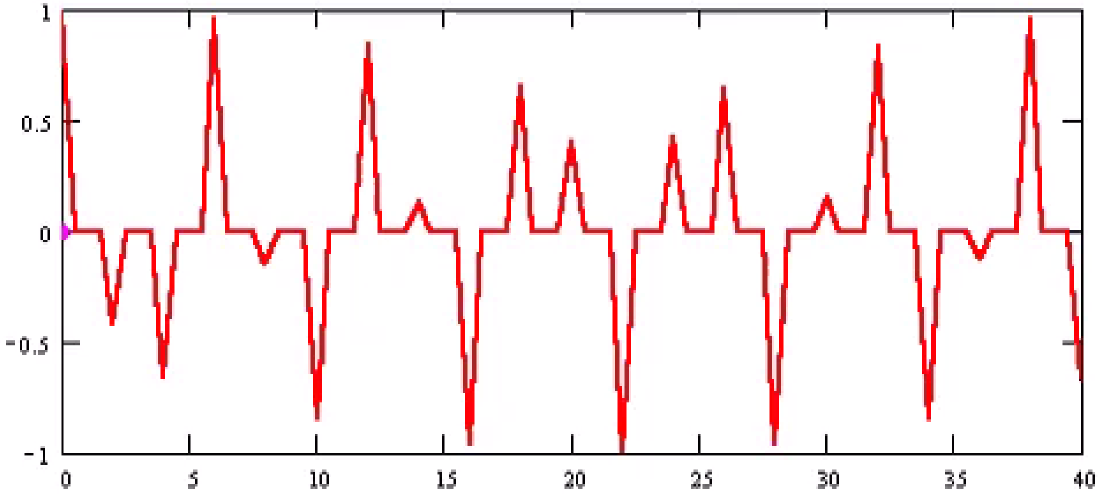
The **synchronous particle** is an **hypothetical particle** propagating along the reference trajectory.

It is used to **synchronize** the time-varying (RF) components.

It **represents the accelerator** and not the beam.

It gives the **nominal arrival time and energy** at a given ***s***, by design.





All the beam particles will be represented with respect to this synchronous particle.

Moving frame

Particle motion is described in a **moving frame**, centered on the orthogonal projection of the particle on the reference trajectory.

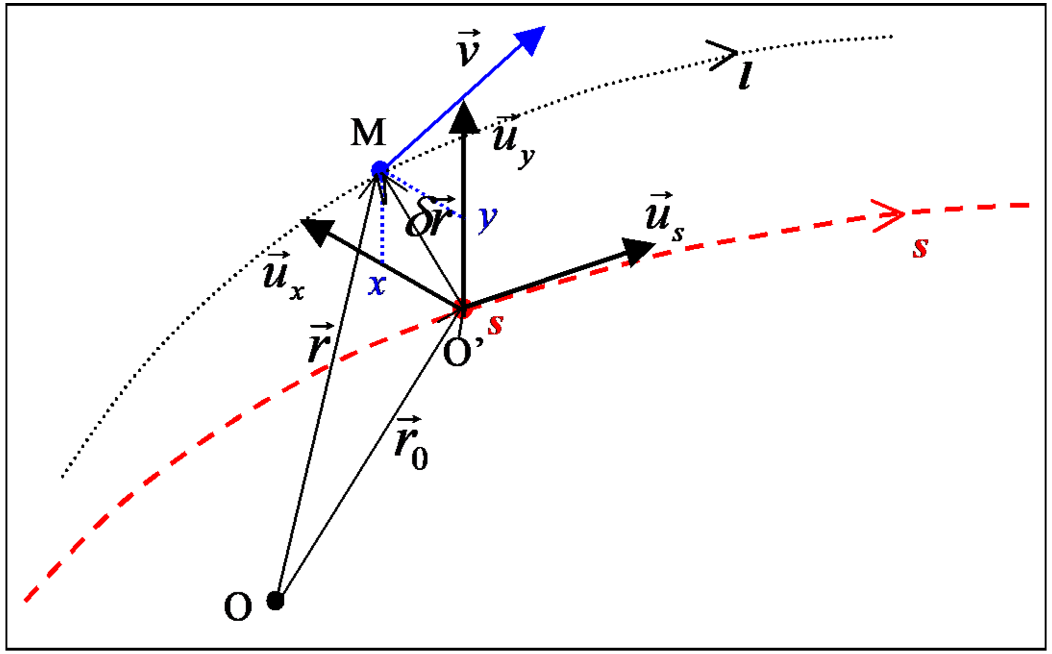
$$\begin{pmatrix} \overrightarrow{u_x} \\ \overrightarrow{u_y} \\ \overrightarrow{u_s} \end{pmatrix}$$

$\overrightarrow{u_s}$: tangent to the reference trajectory

$\overrightarrow{u_x}$: in rotation plan (pointing to the left)

$\overrightarrow{u_y}$: in vertical direction (pointing to the top)

transverse plan



M is the particle point at time *t*

s is absissa of the transverse plan, orthogonal to the reference trajectory containing *M*

In this plan, the particle position is : $\begin{pmatrix} x \\ y \end{pmatrix}$

New particle representation

In this new frame and independent variable, a particle vector becomes:

$$t \rightarrow s$$

$$\downarrow \qquad \downarrow$$

$$\begin{pmatrix} x \\ y \\ z \\ p_x \\ p_y \\ p_z \end{pmatrix} \rightarrow \begin{pmatrix} x \\ y \\ \varphi \\ x' \\ y' \\ K \end{pmatrix}$$

$$\boxed{\vec{P}(s)}$$

- (x, y) are (still) the particle **transverse coordinates**
- $x' = \frac{p_x}{p_z} = \frac{dx}{ds}$ and $y' = \frac{p_y}{p_z} = \frac{dy}{ds}$

  are the particle **transverse slopes**
- $\varphi = 2\pi f_0 \cdot \tau$ is the particle **phase**, the arrival time τ at s normalized wrt a frequency f_0 (generaly RF)
- K is the particle **kinetic energy**

RF acceleration & Longitudinal dynamics

Longitudinal phase-space

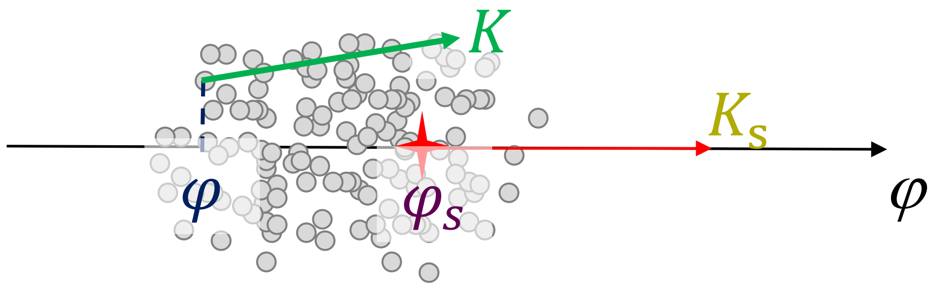
Each beam particle is represented relatively to the **synchronous particle**

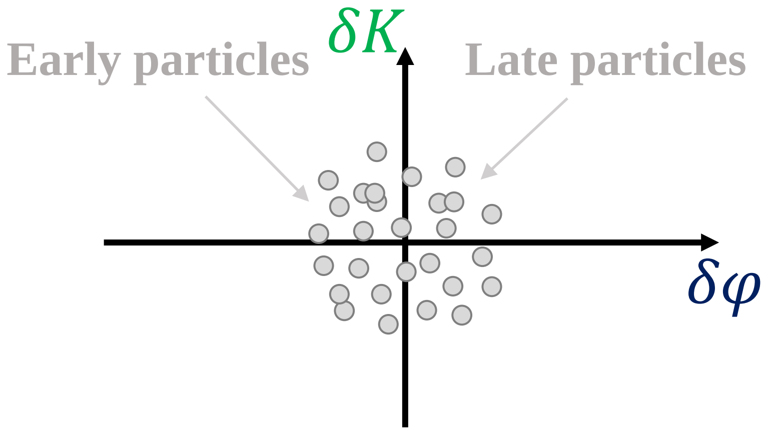
$$\vec{P}_s = \begin{pmatrix} 0 \\ 0 \\ \varphi_s \\ 0 \\ 0 \\ K_s \end{pmatrix}$$

$$\vec{P} = \begin{pmatrix} x \\ y \\ \varphi \\ x' \\ y' \\ K \end{pmatrix}$$

Phase or time: $\delta\varphi = \varphi - \varphi_s$

Energy: $\delta K = K - K_s$





The longitudinal dynamics describes particle trajectories in 2D $\begin{pmatrix} \delta\varphi \\ \delta K \end{pmatrix}$ phase-space

RF cavity

<u>Goal</u> : Give kinetic energy to the beam

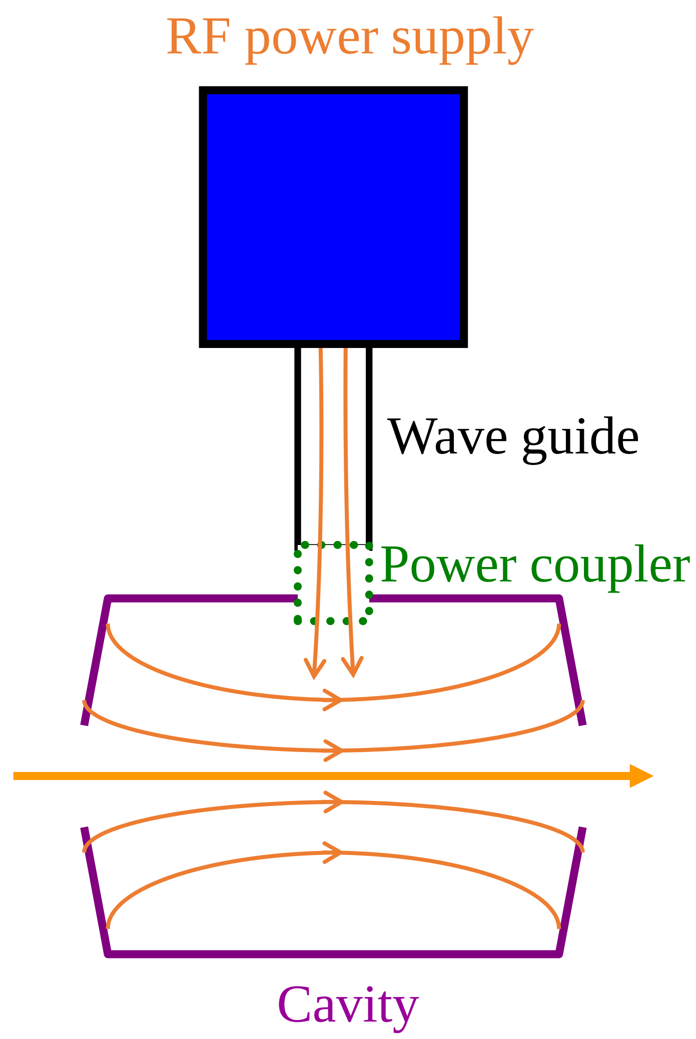
<u>Basic principle</u>

- Conductor enclosing a close volume,

- Maxwell equations + *Boundary conditions* allow possible electromagnetic field E_n/B_n configurations each oscillating with a given frequency f_n : a ***resonant mode***. The field is a weighted superposition of these modes.

- The wanted (accelerating) mode is excited at the good frequency and position from a RF power supply through a power coupler,

-The phase of the electric field is adjusted to accelerate the beam.

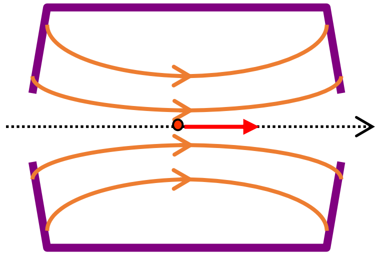


$$\vec{E}(\vec{r},t) = \overrightarrow{E_0}(\vec{r}) \cdot \sin(2\pi f_{RF} t + \phi_0)$$

$$\vec{B}(\vec{r},t) = \overrightarrow{B_0}(\vec{r}) \cdot \cos(2\pi f_{RF} t + \phi_0)$$

Energy gain

Energy gained by a particle in a cavity :



$$\Delta K^{*} = \int_{-\infty}^{\infty} q \cdot E_z(s) \cdot \sin\big(\varphi(s)\big) \cdot ds$$

with : $\varphi(s) = 2\pi f_0 \int_0^s \frac{ds'}{v_z(s')} + \varphi_0$

The field phase when the particle is at position s

$$\Rightarrow \quad \boxed{\Delta K = q \cdot V_0 T(K) \cdot \sin(\phi)}$$

with : $V_0 T(K) = \left| \int_{-\infty}^{\infty} E_z(s) \cdot \mathbf{exp}\big(j \cdot \varphi(s)\big) \cdot ds \right|$

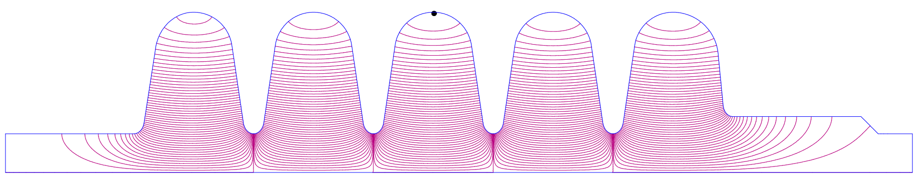
The **cavity effective voltage**

$$\phi = \operatorname{atan}\left(\frac{\int_{-\infty}^{\infty} E_z(s) \cdot \mathbf{sin}\big(\varphi(s)\big) \cdot ds}{\int_{-\infty}^{\infty} E_z(s) \cdot \mathbf{cos}\big(\varphi(s)\big) \cdot ds} \right)$$

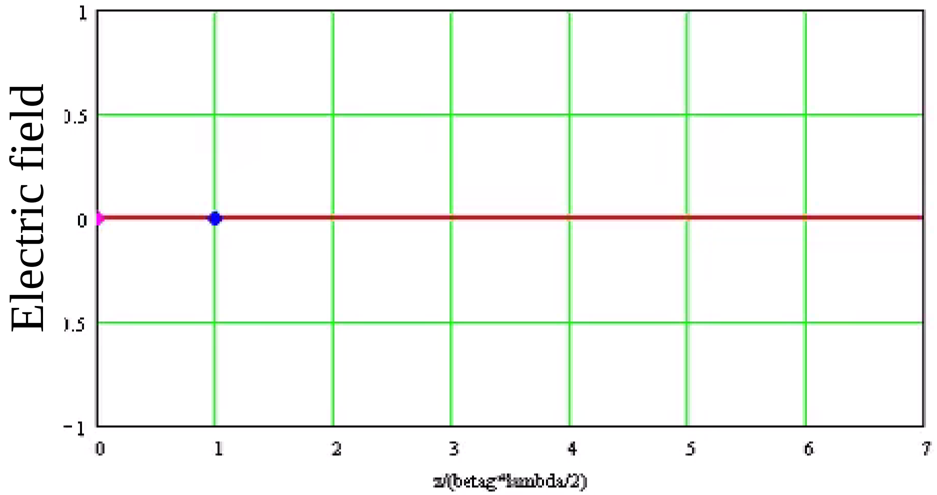
The **particle effective phase** in the cavity

* Sometime (for example linac), one can find : $\Delta K = \int_{-\infty}^{\infty} q \cdot E_z(s) \cdot \mathbf{cos}\big(\varphi(s)\big) \cdot ds$

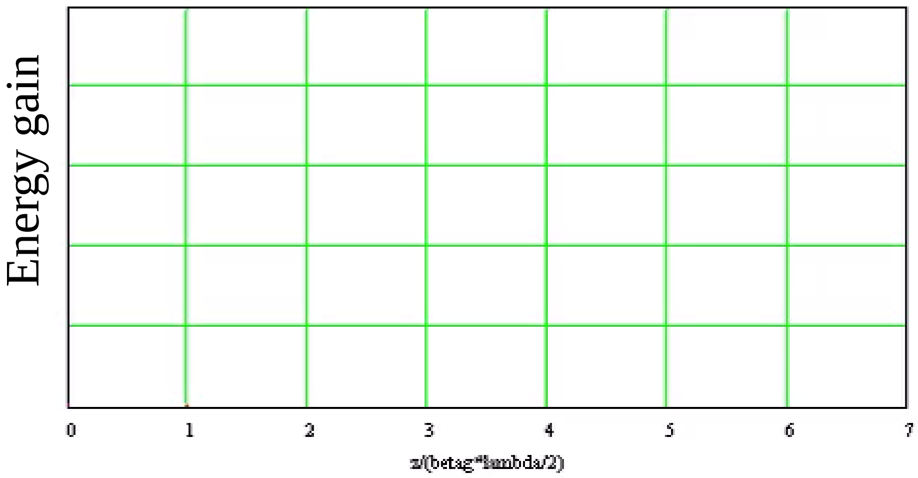
Energy gain in a multicell cavity



$$\Delta K = q \cdot V_0 T(K) \cdot \overset{\overset{1}{\shortparallel}}{\sin(\phi)}$$



- Cavity Oscillating field
- Field seen by an optimal velocity particle
- Field seen by a non-optimal velocity particle



- Hypothetical max energy gain : $q \cdot V_0$
- Energy gained by the optimal velocity particle
- Energy gained by a non-optimal velocity particle

Relative phase change in drift (no bending)

$$\frac{d\delta\varphi}{ds} = \frac{d\varphi}{ds} - \frac{d\varphi_s}{ds} = 2\pi f_0 \cdot \left(\frac{1}{v_z} - \frac{1}{v_s}\right) = -2\pi f_0 \cdot \left(\frac{\delta v_z}{v_s^2}\right)$$

First order simplification : $\frac{\delta K}{K_s} = \frac{K - K_s}{K_s} \ll 1$ $\quad v_z = \frac{v}{1 + x'^2 + y'^2} \approx v$ $\quad \frac{\delta v_z}{v_s} = \frac{v_z - v_s}{v_s} \ll 1$

$$1 - \beta^2 = \gamma^{-2} \quad \rightarrow \quad \beta_s \delta\beta = \frac{\delta\gamma}{\gamma_s^3} \quad \rightarrow \quad \delta v_z = \frac{\delta K}{\gamma_s^3 \beta_s mc}$$

One gets (at first order) :

$$\boxed{\frac{d\delta\varphi}{ds} = -\frac{2\pi}{\lambda_0 mc^2} \cdot \frac{\delta K}{\gamma_s^3 \beta_s^3}}$$

$$\lambda_0 = \frac{c}{f_0}$$

Relative energy change in RF cavity

$$\frac{d\delta K}{ds} = \frac{\vec{v}}{v_s} \cdot \vec{F}(s,\varphi) - \overrightarrow{F_s}(s,\varphi_s)$$

$$\vec{F}(s,\varphi) = q \cdot E_0 T(s) \cdot \sin(\varphi - \varphi_c)$$

$$= q \cdot E_0 T(s) \cdot \sin(\varphi_s - \varphi_c + \delta\varphi)$$

$$\overrightarrow{F_s}(s,\varphi_s) = q \cdot E_0 T(s) \cdot \sin(\varphi_s - \varphi_c)$$

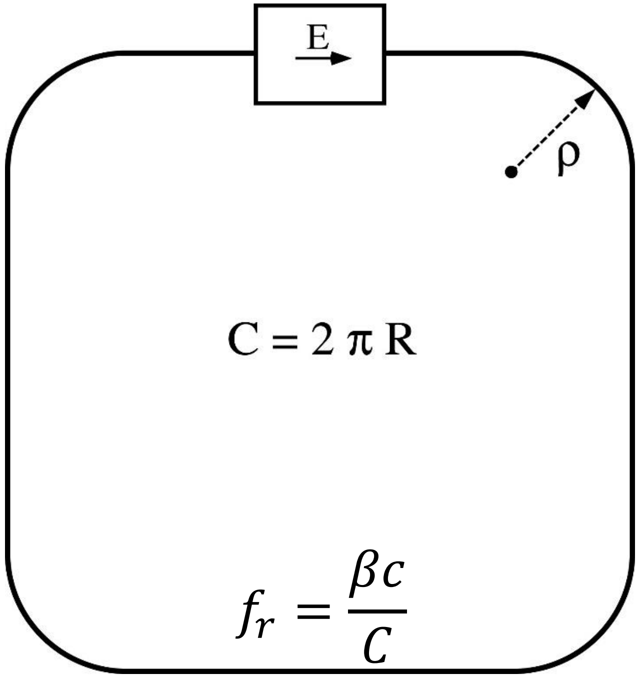
One gets (at first order) :

$$\boxed{\frac{d\delta K}{ds} = q \cdot E_0 T(s) \cdot \left(\sin(\varphi_s^* + \delta\varphi) - \sin(\varphi_s^*)\right)}$$

$$\varphi_s^* = \varphi_s - \varphi_c$$

φ_s^* is the average phase seen by synchronous particle in each cavity

Specificity for circular machine



The beam revolution frequency.

$$\boxed{\eta = \frac{\delta f_r / f_r}{\delta p / p}}$$

is the relative variation of revolution frequency with respect to the relative momentum

$$f_r = \frac{\beta c}{C} \quad \Rightarrow \quad \frac{\delta f_r}{f_r} = \frac{\delta \beta}{\beta} - \frac{\delta C}{C}$$

$$\Rightarrow \quad \boxed{\eta = \gamma^{-2} - \alpha}$$

$$\boxed{\alpha = \frac{\delta C / C}{\delta p / p}}$$

is the ***momentum compaction***

It is specific to circular machine (α =0 in linac)

- $\eta > 0$ **Velocity dominated** : A higher energy particle turns faster → linacs and low energy synchrotrons.
- $\eta < 0$ **Trajectory dominated :** A higher energy particle turns slower → high energy synchrotrons.

Specificity for circular machine

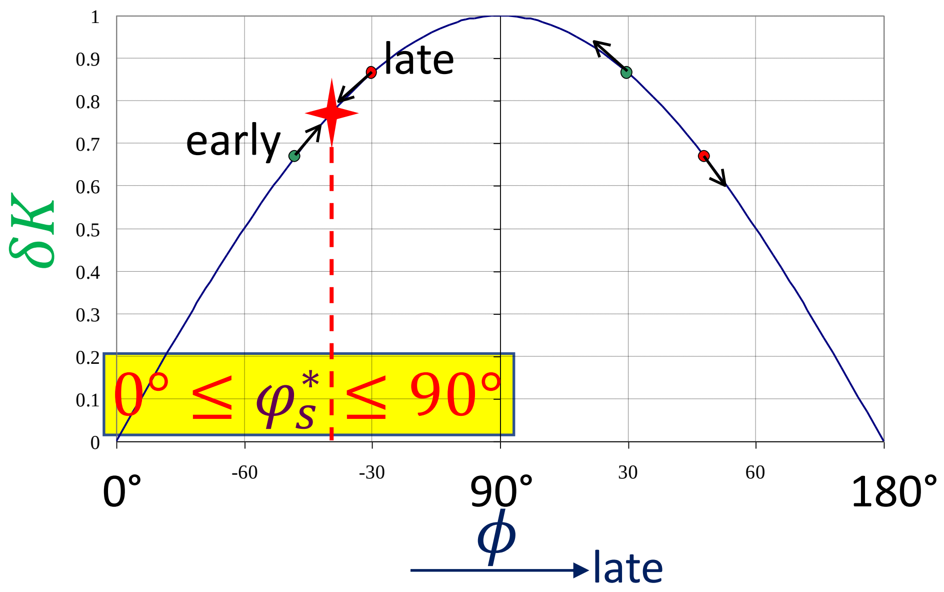
In cavity, the electric field is oscillating with time :

$$E_z = E_0 \cdot sin(\varphi)$$

!! In some convention, one can have $E_z = E_0 \cdot cos(\varphi)$

$$\delta K = q \cdot V_0 T(s) \cdot (\sin(\varphi_s^* + \delta\varphi) - \sin(\varphi_s^*))$$

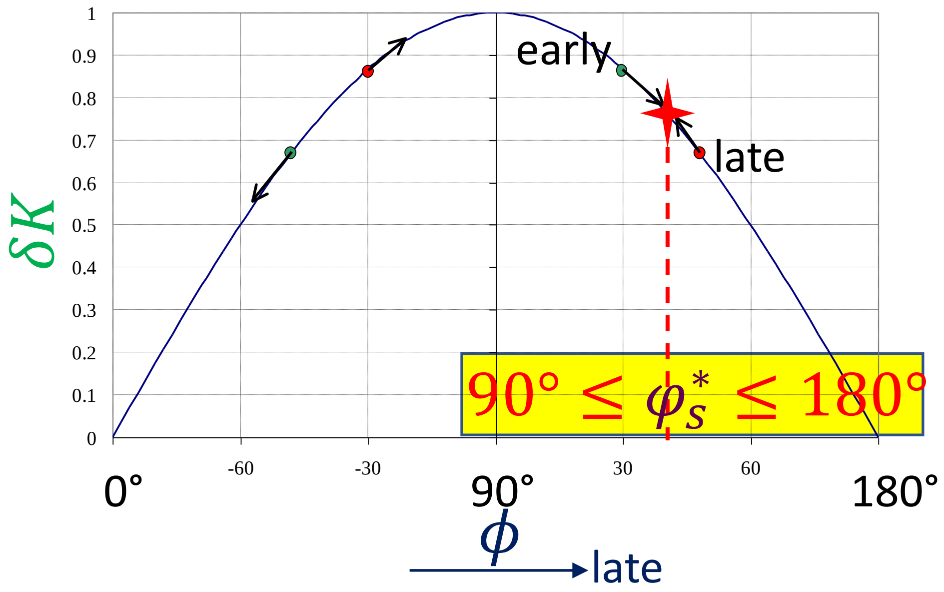
Case 1: **Velocity dominated** $\eta > 0$



The late particle shall gain more energy

(to increase its speed)

Case 2: **Trajectory dominated** $\eta < 0$



The early particle shall gain more energy

(to elongate its trajectory)

Correction in circular machine

In linac

$$\frac{d\delta\varphi}{ds} = -\frac{2\pi}{\lambda_0 mc^2} \cdot \frac{\delta K}{\gamma_s^3 \beta_s^3}$$

+

In general

$$\eta = \frac{1}{\gamma_s^2} - \alpha$$

➔ Replace $\frac{1}{\gamma_s^2}$ by $\eta = \frac{1}{\gamma_s^2} - \alpha$

➔ $$\boxed{\frac{d\delta\varphi}{ds} = -\frac{2\pi\,\eta}{\lambda_0 mc^2} \cdot \frac{\delta K}{\gamma_s \beta_s^3}}$$

Longitudinal motion equation

$$\begin{cases} \dfrac{d\delta\varphi}{ds} = -\dfrac{2\pi\,\eta}{\lambda_0 mc^2}\cdot\dfrac{\delta K}{\gamma_s\beta_s^3} & = -\dfrac{\partial H(\delta\varphi,\delta K)}{\partial\delta K} \\ \dfrac{d\delta K}{ds} = q\cdot E_0T(s)\cdot\left(\sin(\varphi_s^*+\delta\varphi)-\sin(\varphi_s^*)\right) & = \dfrac{\partial H(\delta\varphi,\delta K)}{\partial\delta\varphi} \end{cases}$$

$H(\delta\varphi,\delta K)$ is the motion Hamiltonian :

$$H(\delta\varphi,\delta K) = \frac{\pi\,\eta}{\lambda_0 mc^2}\cdot\frac{\delta K^2}{\gamma_s\beta_s^3}$$
$$-q\cdot E_0T(s)\cdot\left(\sin(\varphi_s^*)\cdot(\delta\varphi-\sin(\delta\varphi))-\cos(\varphi_s^*)\cdot(1-\cos(\delta\varphi))\right)$$

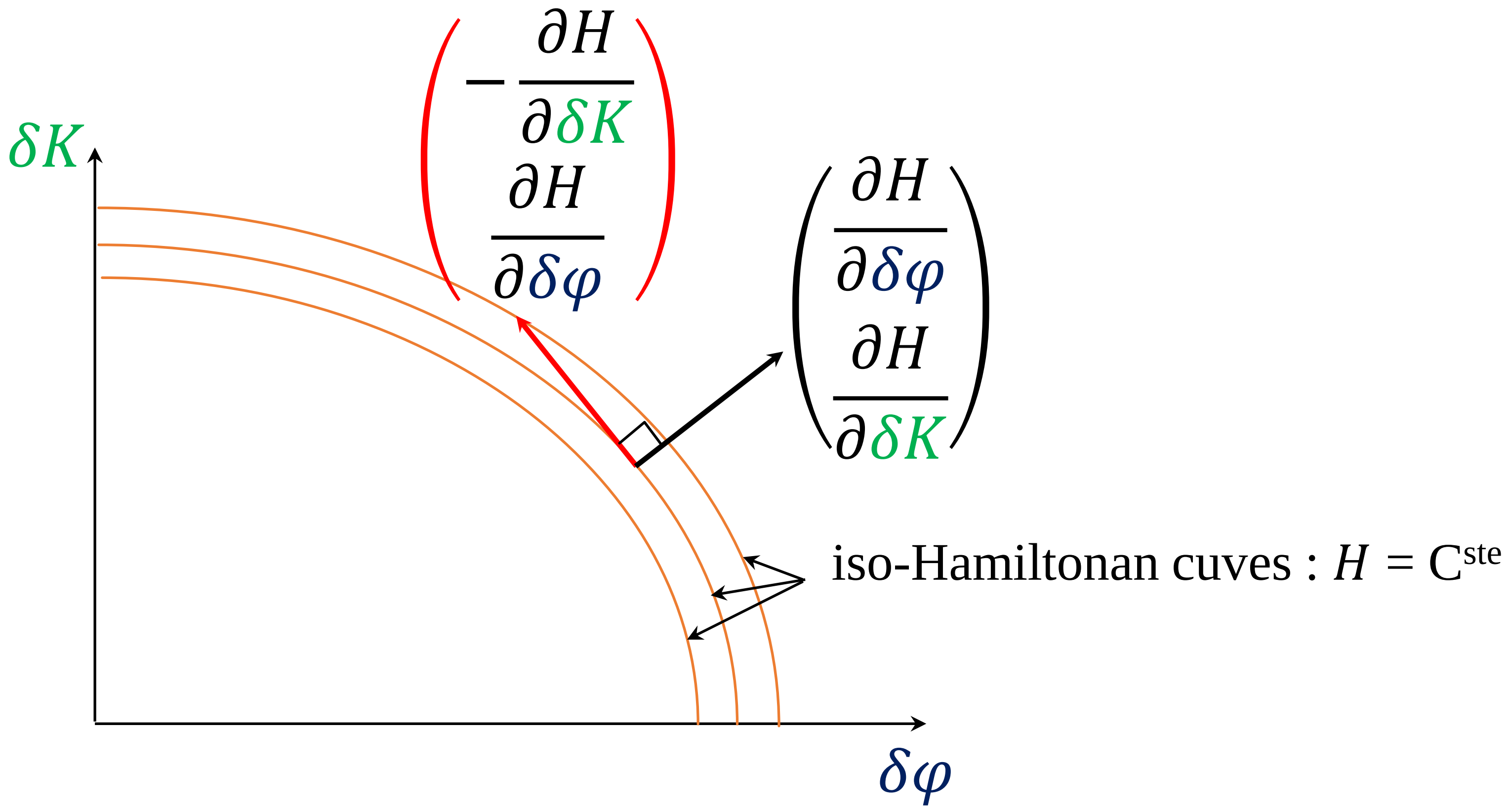
Hamiltonian

The particle motion can be described using a function of phase and energy:
the motion ***Hamiltonian*** $H(\delta\varphi, \delta K)$

Hamiltonian definition:

$$\frac{d\delta\varphi}{ds} = -\frac{\partial H(\delta\varphi, \delta K)}{\partial \delta K}$$

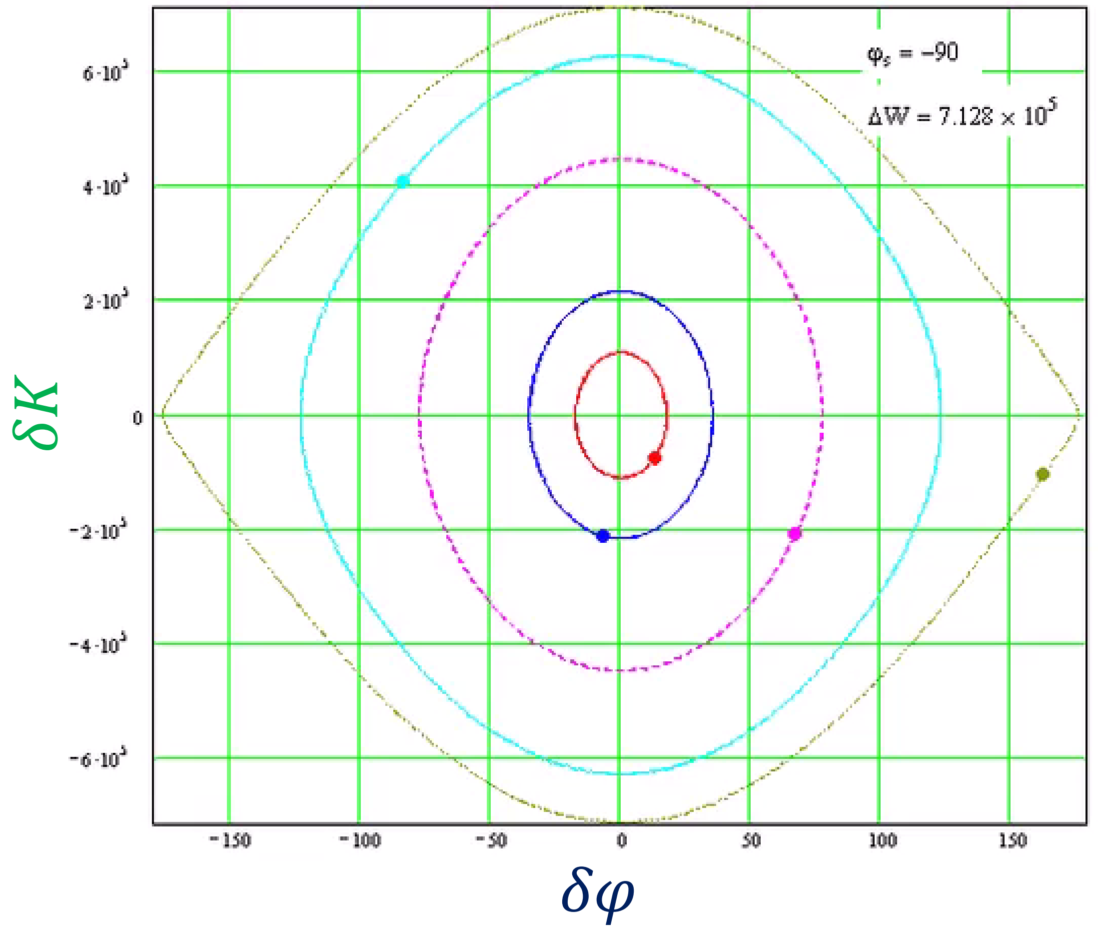
$$\frac{d\delta K}{ds} = \frac{\partial H(\delta\varphi, \delta K)}{\partial \delta\varphi}$$



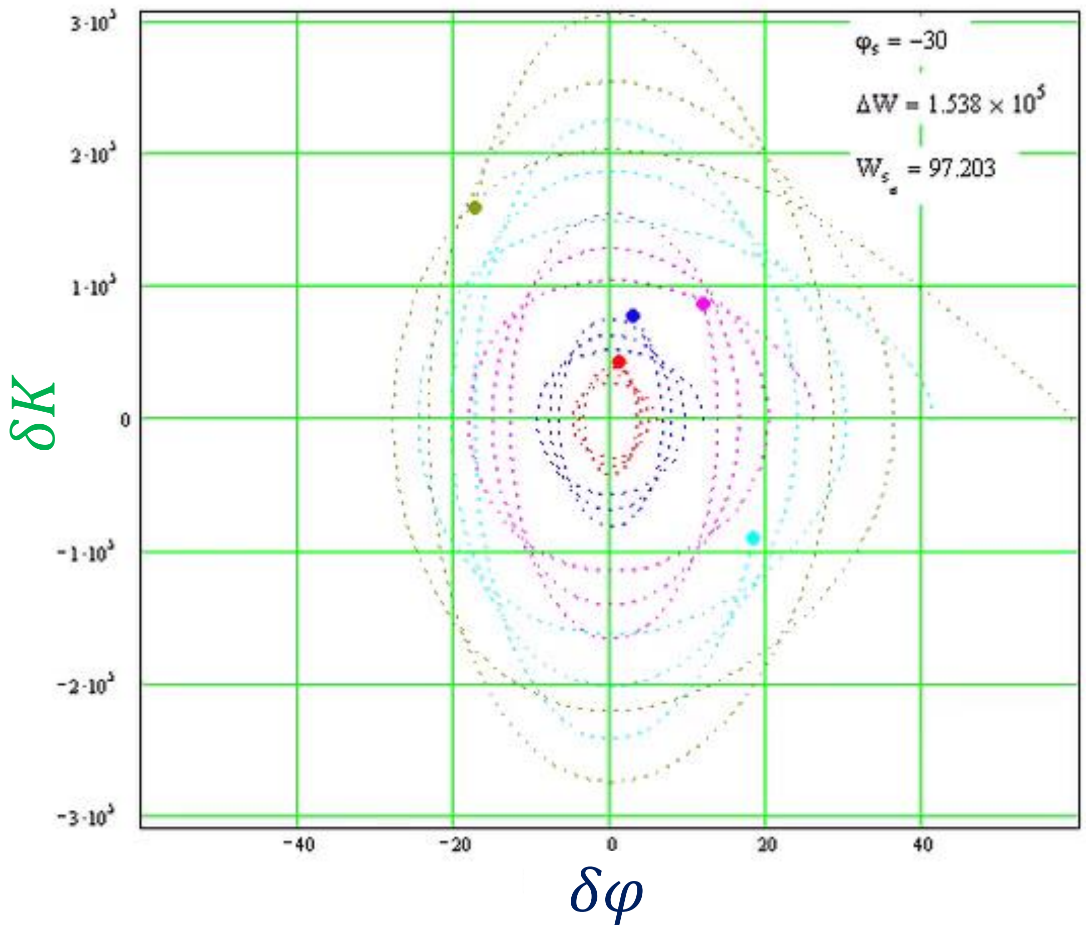
The Hamiltonian $H(\delta\varphi, \delta K)$ gives the particle trajectory in phase-space

Phase-space trajectories

$\varphi_s = 0°$: no acceleration (storage), maximum stability



$\varphi_s = 60°$: acceleration, but reduced stability



Linear dynamics

Particle motion in an accelerator

$$\vec{P} = \begin{pmatrix} x \\ y \\ \varphi \\ x' \\ y' \\ K \end{pmatrix} \qquad \vec{P_s} = \begin{pmatrix} 0 \\ 0 \\ \varphi_s \\ 0 \\ 0 \\ K_s \end{pmatrix}$$

$$\underbrace{\qquad\qquad\qquad}_{\overrightarrow{\delta P} = \vec{P} - \vec{P_s}}$$

+

$$\frac{d\vec{p}}{dt} = q \cdot \left(\vec{v} \times \vec{B} + \vec{E}\right) = \vec{F}$$

$$\vec{p} = mc \cdot \gamma\vec{\beta}$$

$$\frac{ds}{dt} = \frac{\beta_z c}{(1 + h(K, s) \cdot x)}$$ (p. 13)

Motion equations

$$\frac{d\vec{\boldsymbol{P}}}{ds}\left\{\begin{aligned}
\frac{dx}{ds} &= x' \\
\frac{dy}{ds} &= y' \\
\frac{d\varphi}{ds} &= \frac{2\pi}{\lambda_0}\frac{(1+h(K,s)\cdot x)}{\beta_z c} \\
\frac{dx'}{ds} &= h_s(s) + (1+h(K,s)\cdot x)\cdot\frac{F_x(s)}{mc^2\gamma\beta_z^2} + x'\cdot\frac{d\gamma\beta_z}{\gamma\beta_z ds} \\
\frac{dy'}{ds} &= (1+h(K,s)\cdot x)\cdot\frac{F_y(s)}{mc^2\gamma\beta_z^2} + y'\cdot\frac{d\gamma\beta_z}{\gamma\beta_z ds} \\
\frac{dK}{ds} &= (1+h(K,s)\cdot x)\cdot\left(\vec{v}\cdot\vec{F}(s)\right)
\end{aligned}\right.$$

Centrifugal force

EM force

Acceleration damping

With :

$$\gamma = 1 + \frac{K}{mc^2}$$

$$\beta_z = \sqrt{\frac{1-\gamma^{-2}}{1+x'^2+y'^2}}$$

$$\vec{F} = q\cdot\left(\vec{E}+\vec{v}\times\vec{B}\right)$$

Linearisation

These equations are :

Non-linear $F_x(\ldots, x, x^2, \ldots)$ $F_y(\ldots, y, y^2, \ldots)$

Coupled $F_x(\ldots, y, \varphi, K, \ldots)$ $F_y(\ldots, x, \varphi, K, \ldots)$

Damped $\frac{dx'}{ds} \propto x' \cdot \frac{d\gamma\beta_z}{\gamma\beta_z ds}$ $\frac{dy'}{ds} \propto y' \cdot \frac{d\gamma\beta_z}{\gamma\beta_z ds}$

In order to solve the particle dynamics, one shall compute (generally numerically) these complex equations.

Nevertheless the computation can by simplified (and understood) by **linearising** the forces around the synchronous particle and using a "simple" matrix formalism.

$$\overrightarrow{\boldsymbol{\delta P}} = \vec{P} - \overrightarrow{P_s}$$

Matrix formalism

Linearised differential motion equations : $\frac{d\delta P_i}{ds} = \sum_{j=1}^{6} \frac{\partial \delta P_i}{\partial \delta P_j}(s) \cdot \delta P_j$

$$\rightarrow \quad \delta P_i(s + ds) = \delta P_i(s) + \sum_{j=1}^{6} \frac{\partial \delta P_i}{\partial \delta P_j}(s) \cdot ds \cdot \delta P_j\,(s)$$

$$\rightarrow \quad \begin{pmatrix} x \\ y \\ \delta\varphi \\ x' \\ y' \\ \delta K \end{pmatrix}_{s+ds} = \underbrace{\left(I + \left[\frac{\partial \delta P_i}{\partial \delta P_j}(s) \right] \cdot ds \right)}_{6\times 6 \text{ Matrix}} \cdot \begin{pmatrix} x \\ y \\ \delta\varphi \\ x' \\ y' \\ \delta K \end{pmatrix}_{s}$$

Transfer matrix

From an abscissa s_1 to an abscissa s_2:

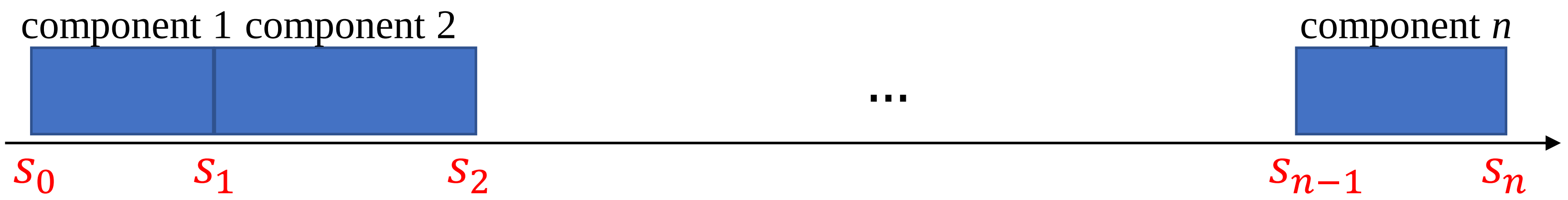
$$\begin{pmatrix} x \\ y \\ \delta\varphi \\ x' \\ y' \\ \delta K \end{pmatrix}_{s_2} = [T(s_2 \leftarrow s_1)] \cdot \begin{pmatrix} x \\ y \\ \delta\varphi \\ x' \\ y' \\ \delta K \end{pmatrix}_{s_1}$$

$[T(s_2 \leftarrow s_1)]$ is the transfer matrix from s_1 to s_2.

Each accelerator element can be modeled (at first order) by a transfer matrix.

Accelerator matrix

For an accelerator from an abscissa s_0 to an abscissa s_n



$$\begin{pmatrix}x\\ y\\ \delta\varphi\\ x'\\ y'\\ \delta K\end{pmatrix}_{s_n} = \underbrace{[T(s_n \leftarrow s_{n-1})]\cdot\ldots\cdot\underbrace{[T(s_2 \leftarrow s_1)]\cdot\underbrace{[T(s_1 \leftarrow s_0)]}_{\text{from } s_0 \text{ to } s_1}}_{\text{from } s_0 \text{ to } s_2}}_{\text{from } s_0 \text{ to } s_n}\cdot\begin{pmatrix}x\\ y\\ \delta\varphi\\ x'\\ y'\\ \delta K\end{pmatrix}_{s_0}$$

Periodic transport

Periodic uncoupled focusing

In the highest simplification level, the external force along direction w (x, y or $\delta\varphi$) can be considered **periodic**, **linear**, **uncoupled** and **undamped** over one period :

Hill equation : $\frac{d^2 w}{ds^2} + k_w(s) \cdot w = 0$ $\quad k_w(s+S) = k_w(s)$

Giving : $w(s) = \sqrt{\varepsilon \cdot \beta_{wm}(s)} \cdot \cos(\mu(s))$ $\quad \varepsilon$ constant,

with: $\mu(s) = \mu_0 + \int_{s_0}^{s} \frac{ds}{\beta_{wm}(s)}$ and: $\beta_{wm}(s+S) = \beta_{wm}(s)$

In the (w, w') phase-space, the particle is moving on an ellipse of equation :

$$\gamma_{wm}(s) \cdot w^2 + 2 \cdot \alpha_{wm}(s) \cdot ww' + \beta_{wm}(s) \cdot w'^2 = \varepsilon$$

with : $\alpha_{wm}(s) = -\frac{1}{2} \cdot \frac{d\beta_{wm}}{ds}$ and $\gamma_{wm}(s) = \frac{1 + \alpha_{wm}(s)^2}{\beta_{wm}(s)}$ Courant-Snyder parameters

The *phase advance* of the particle in a lattice is then : $\sigma = \mu(s+S) - \mu(s)$

Courant-Snyder parameters from matrix

The 2D (decoupled) transfer matrix of one lattice from s to $s+S$ can be written :

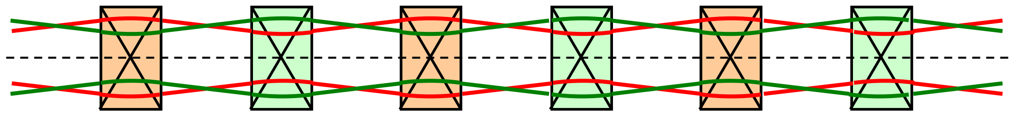
$$[T_w(s+S \leftarrow s)] = \begin{pmatrix} \cos\sigma + \alpha_{wm}(s)\cdot\sin\sigma & \beta_{wm}(s)\cdot\sin\sigma \\ -\gamma_{wm}(s)\cdot\sin\sigma & \cos\sigma - \alpha_{wm}(s)\cdot\sin\sigma \end{pmatrix}$$

σ is the lattice phase advance and α_{wm}, β_{wm} and γ_{wm} are the Courant-Snyder at s :

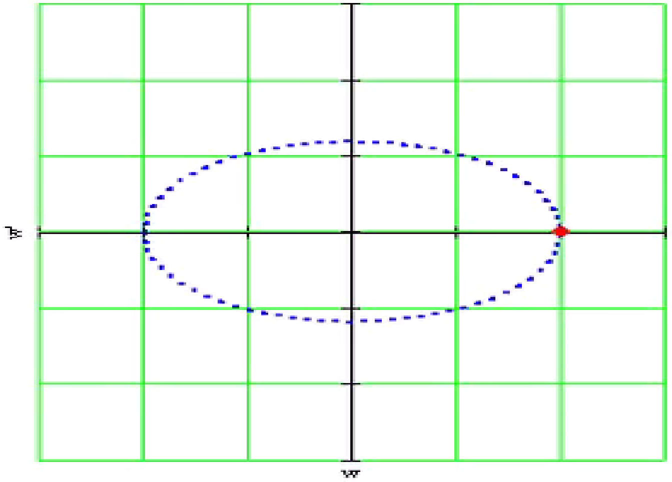
$$\boxed{\left\{\begin{aligned} \cos\sigma &= \frac{1}{2}\cdot\left(T_{w_{11}} + T_{w_{22}}\right) \\ \beta_{wm} &= \frac{T_{w_{12}}}{\sin\sigma} \\ \gamma_{wm} &= -\frac{T_{w_{21}}}{\sin\sigma} \\ \alpha_{wm} &= \frac{T_{w_{11}} - T_{w_{22}}}{2\cdot\sin\sigma} \end{aligned}\right.}$$

Example : motion in FODO lattice

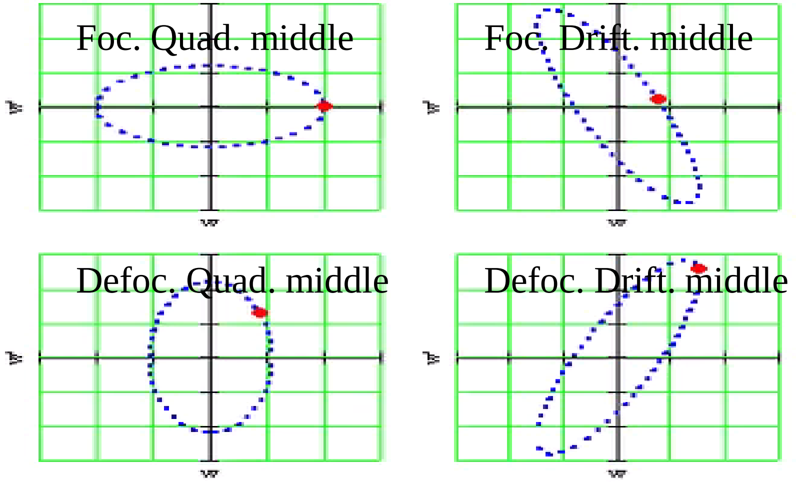
- Particle
- Particle ellipse

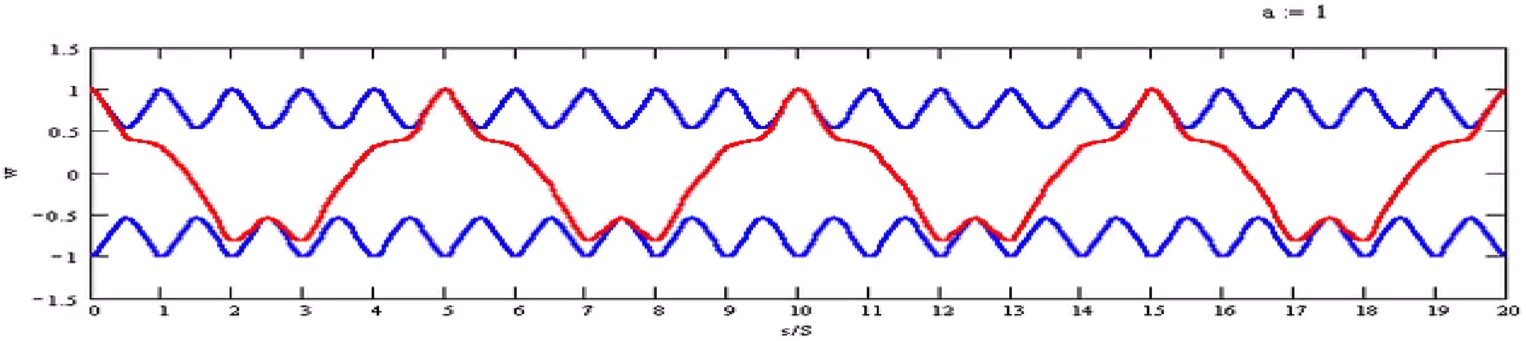


Phase-space trajectory



Phase-space periodic looks





Particle trajectory   Particle ellipse maximum size

Ellipsis' shapes are given by courant-Snyder parameters

- Period after period, the particle is turning by an angle given by the phase advance (here 72°)

Beam

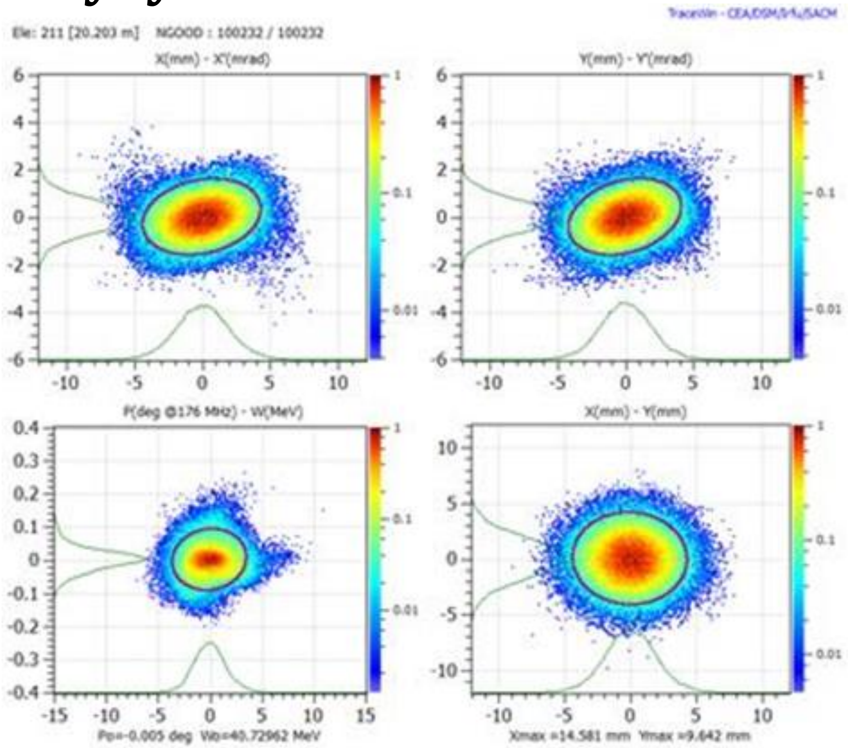
Beam distribution

A **beam** is a set of charged particles

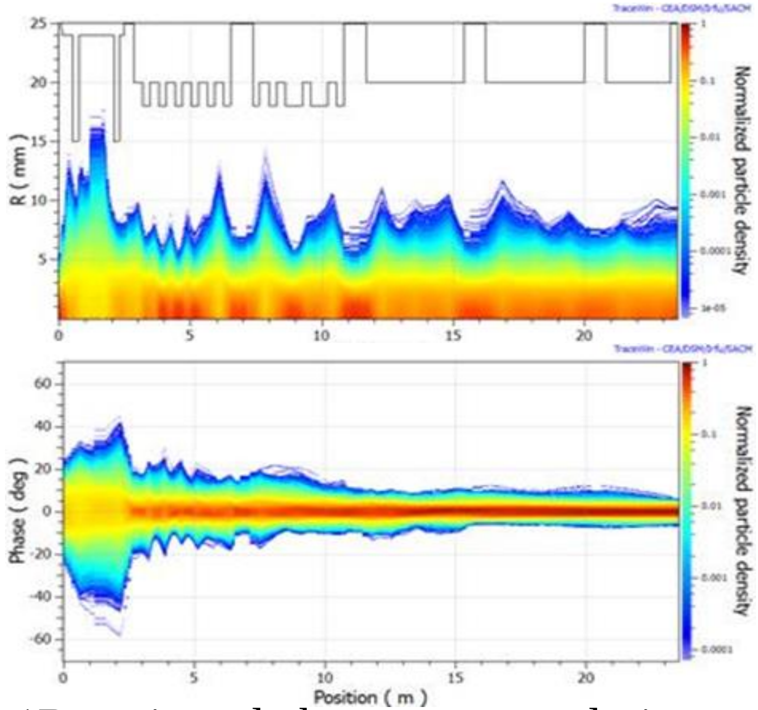
It is represented by a distribution in 6D phase-space : $f(\vec{P}, s)$

$f(\vec{P}, s) \cdot d\vec{P}$ is the average number of particles in $d\vec{P}$ phase-space cell.

$\int\int f(\vec{P}, s) \cdot d\vec{P} = N(s)$ is the total number of particles in the beam



2D projected phase-space distribution



1D projected phase-space evolution

- Each dot is a (macro) particle
- The colour represents the particle density
- Statistics properties can be calculated from these distributions

Beam statistics simplification

100 mA → N = 6 10^{17} charges/s

1 nC → N = 6 10^{9} charges

A beam should be represented by 6 times this huge number of particles.

Moreover, the exact 6D positions of each particle cannot be known precisely

The **average value** of a quantity Q, $\langle Q \rangle$, in the beam is defined by :

$$\langle Q \rangle(s) = \frac{1}{N} \cdot \int \int f(\vec{P}, s) \cdot Q(\vec{P}) \cdot d\vec{P}$$

Q can be a number, a vector, a matrix… the average applies to all components of Q.

→ The beam distribution can be simplified by its **simplified by its momenta**

Beam momenta

<u>Order 0</u> : $Q(s) = N(s) \cdot \langle q \rangle$, the bunch **charge** [C] or beam **current** [C/s]

<u>Order 1</u> : $\langle \vec{\boldsymbol{P}} \rangle (s)$, the beam **center of mass** 6D vector

<u>Order 2</u> : $[\Sigma](s) = \langle \vec{\boldsymbol{P}}(s) \cdot \vec{\boldsymbol{P}}(s)^T \rangle$, the beam 6×6 **sigma matrix**

$$\rightarrow \quad [\Sigma] = \left[\begin{pmatrix} \langle (P_1 - \langle P_1 \rangle) \cdot (P_1 - \langle P_1 \rangle) \rangle & \cdots & \langle (P_1 - \langle P_1 \rangle) \cdot (P_6 - \langle P_6 \rangle) \rangle \\ \vdots & \ddots & \vdots \\ \langle (P_6 - \langle P_6 \rangle) \cdot (P_1 - \langle P_1 \rangle) \rangle & \cdots & \langle (P_6 - \langle P_6 \rangle) \cdot (P_6 - \langle P_6 \rangle) \rangle \end{pmatrix} \right]$$

→ Order 0-2 are the most useful models of the beam

RMS and 2D Ellipsis model

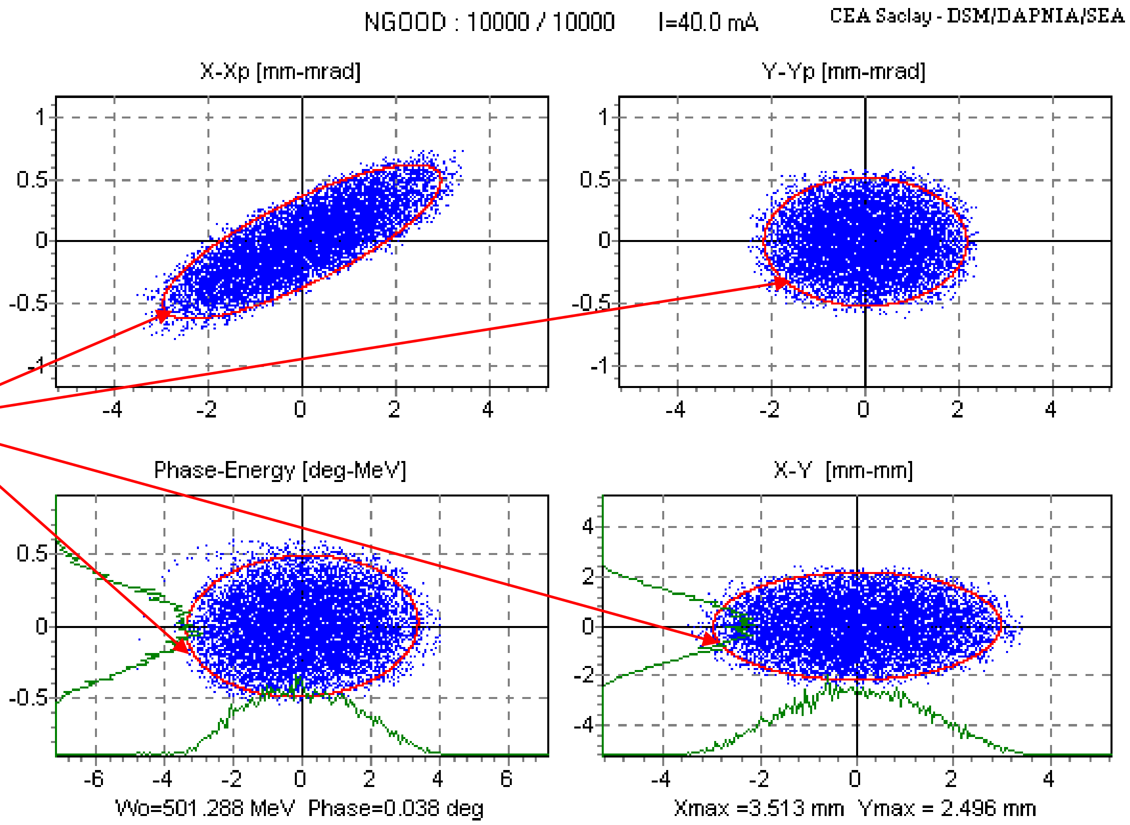
$$\widetilde{P_i} = \sqrt{\langle (P_i - \langle P_i \rangle) \cdot (P_i - \langle P_i \rangle) \rangle}$$ i **RMS sizes**

$$\widetilde{P_{ij}} = \langle (P_i - \langle P_i \rangle) \cdot (P_j - \langle P_j \rangle) \rangle$$ i-j **RMS coupling**

$$\widetilde{\varepsilon_{ij}} = \sqrt{\widetilde{P_i}^2 \cdot \widetilde{P_j}^2 - \widetilde{P_{ij}}^2}$$ **RMS emittance**

$$\widetilde{\gamma_j} \cdot a_i^2 + 2 \cdot \widetilde{\alpha_{ij}} \cdot a_i \cdot a_j + \widetilde{\beta_i} \cdot a_j^2 = n \cdot \widetilde{\varepsilon_{ij}}$$

is the equation of an ellipsis fitting the distribution



$$\widetilde{\beta_i} = \frac{\widetilde{P_i}^2}{\widetilde{\varepsilon_{ij}}} \qquad \widetilde{\gamma_j} = \frac{\widetilde{P_j}^2}{\widetilde{\varepsilon_{ij}}} \qquad \widetilde{\alpha_{ij}} = \frac{\widetilde{P_{ij}}}{\widetilde{\varepsilon_{ij}}}$$

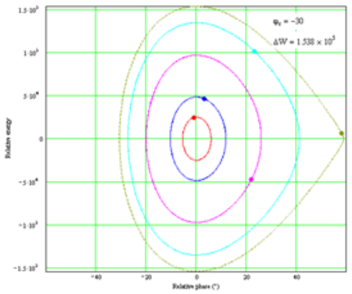
are the **Twiss parameters** of the beam distribution

Matching

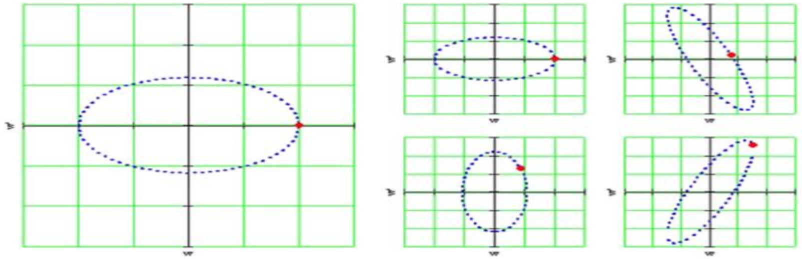
Accelerator vs Beam

Accelerator

In **non-linear forces**, particles are moving in 6D phase-space following curves where motion Hamiltonian is (locally) constant

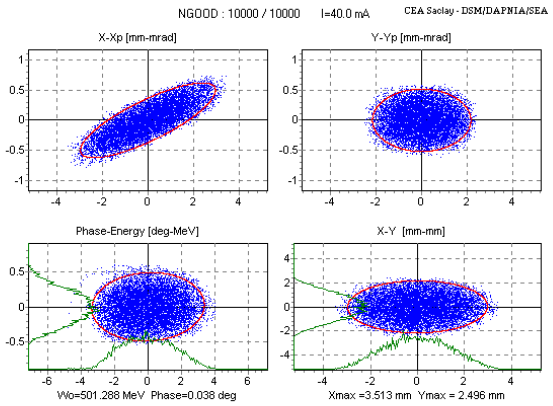


In **periodic linearised forces**, these curves are oscillating 6D ellipsoids



Beam

A beam can be described by a **sigma matrix** whose coefficients correspond to a description of a 6D ellipsoid



Matched beam

The beam is matched to the accelerator when its sigma matrix fits the ellipsoid describing the particle linearised motion.

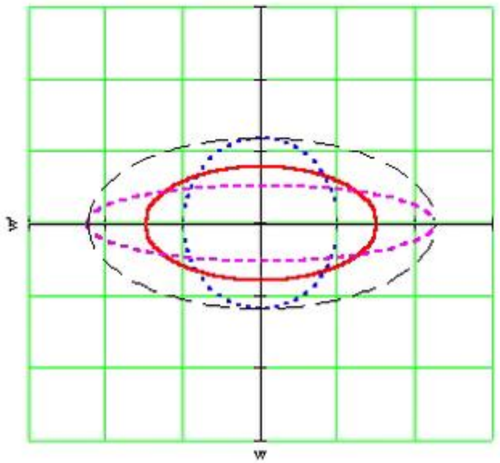
beam ↓ accelerator ↓

$$\widetilde{\beta_w} = \beta_{wm}$$

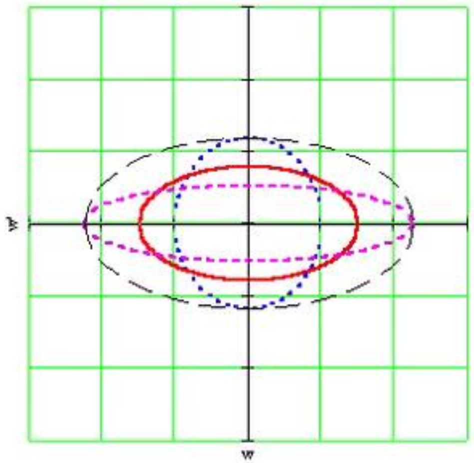
$$\widetilde{\alpha_w} = \alpha_{wm}$$

$$\widetilde{\gamma_w} = \gamma_{wm}$$

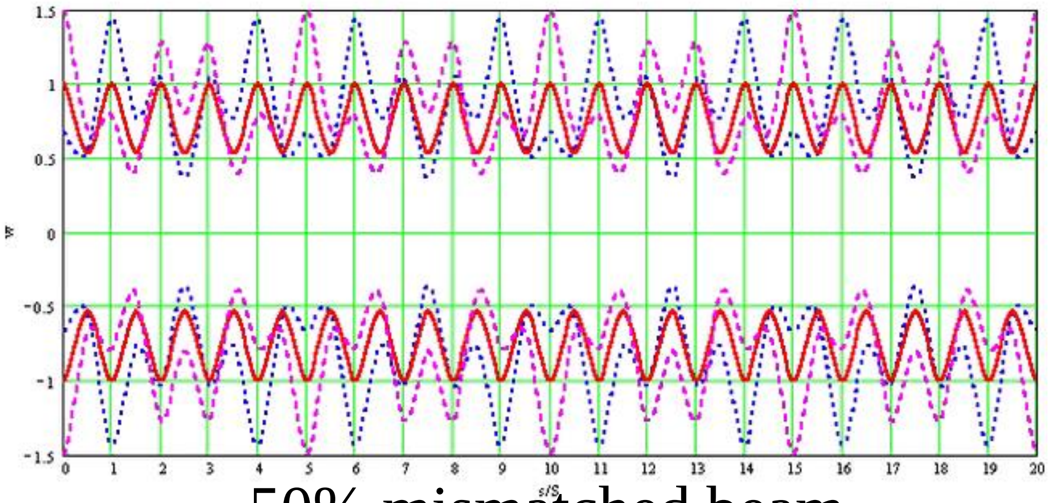
Phase-space trajectory



Phase-space periodic looks



- Matched beam
- Bigger input beam
- Smaller input beam
- Phase-space scanned by the mismatched beams

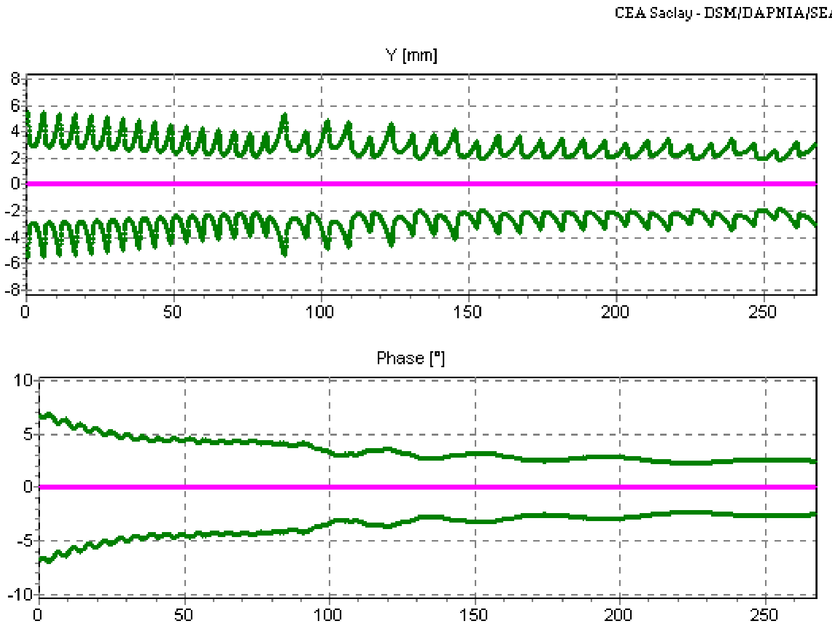


50% mismatched beam

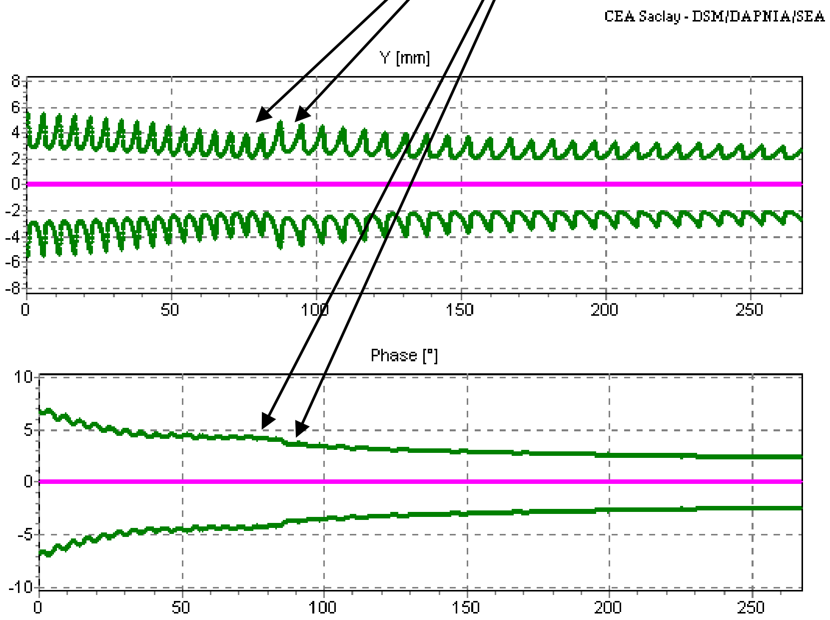
Beam matching

The matching is done between sections by **changing focusing force** with quadrupoles (transverse) and cavities (longitudinal).

Calculations are made with « **envelope codes** » where the beam is modelled by its rms dimensions. This type of code calculates automatically the focusing strength that match the beam.



Non matched beam



Matched beam

Mismatching

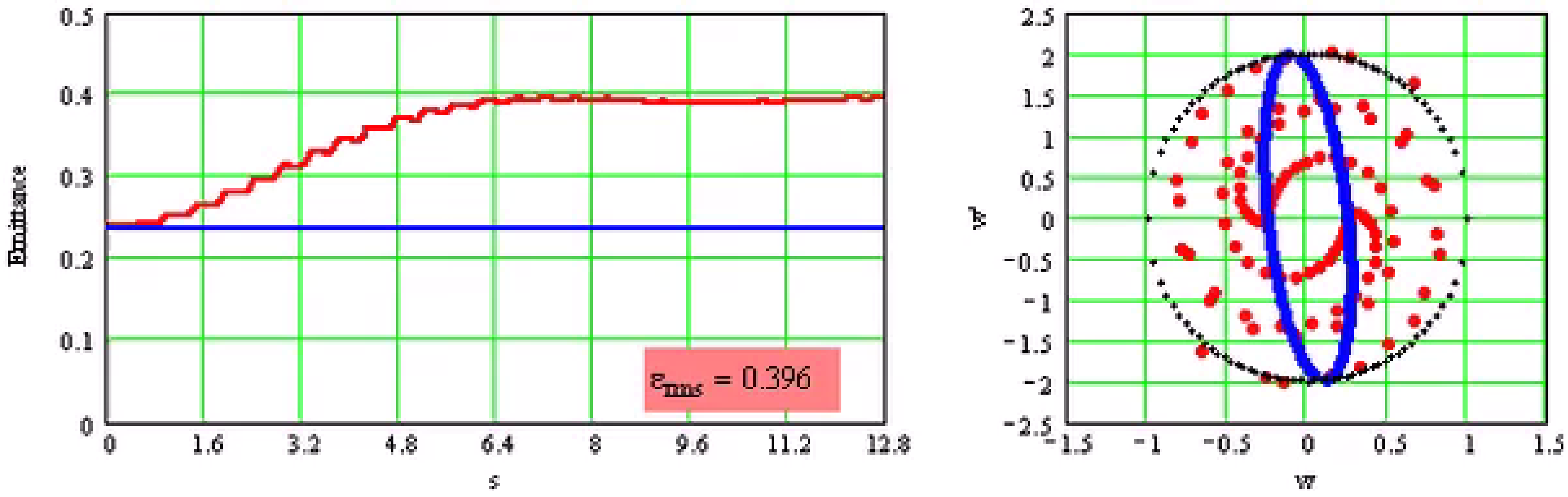
Non linear force

When the confinement force is non linear (multipole, longitudinal, space-charge), the particle oscillation period depends on the oscillation amplitude A.

$$\frac{d^2 w}{ds^2} + k_w(s, w) \cdot w = 0$$

This phenomenon is known as the tune spread.

Particle do not rotate at the same speed in the phase-space : possible filamentation



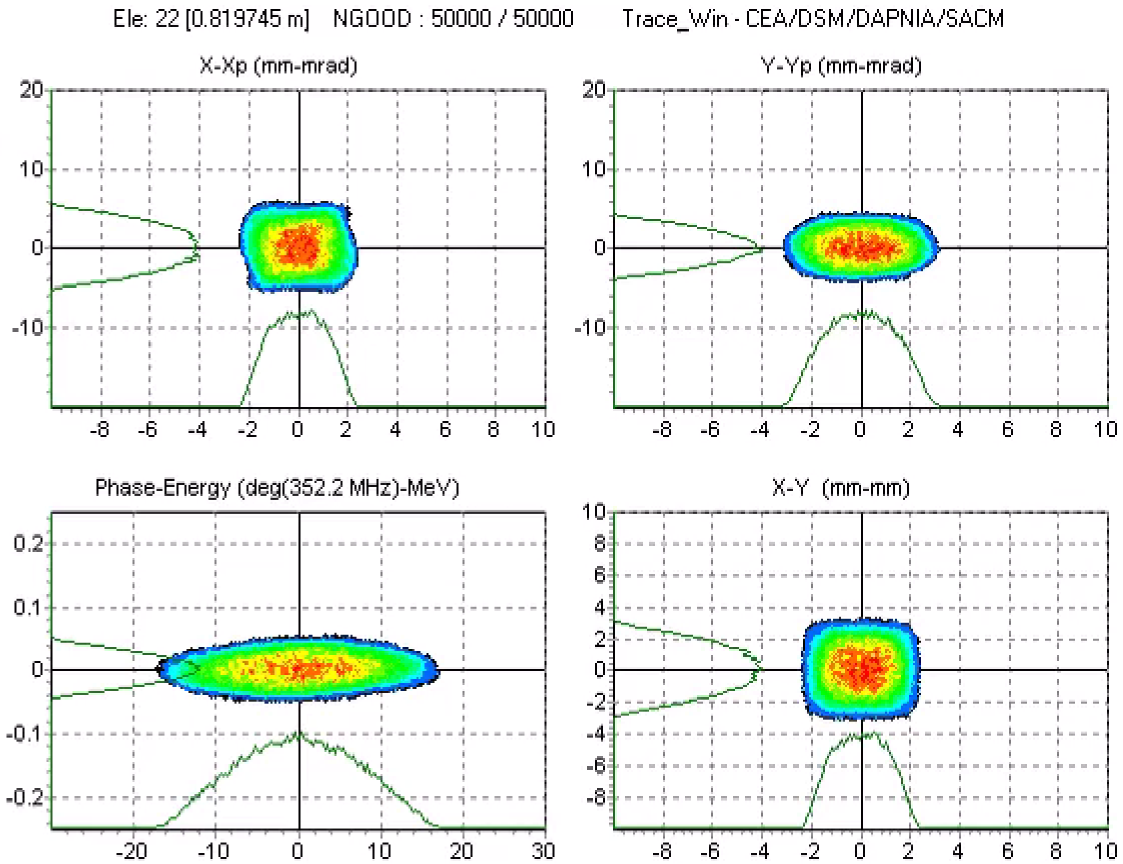
Linear force — Non linear force

Emittance growth

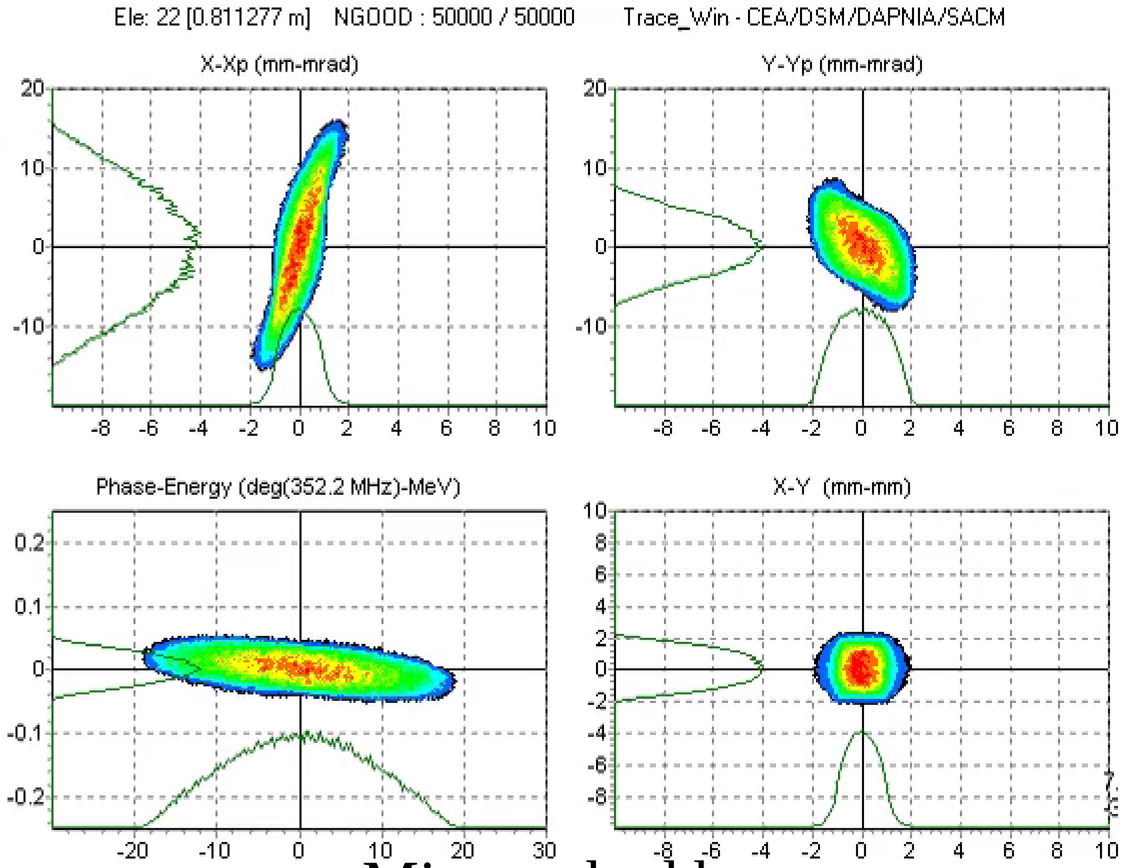
Mismatching :

Emittance growth & Halo formation through :

- non linear-forces (external or space-charge),
- resonance (out of the scope here) of some particle motion with core oscillation.



Matched beam



Mismatched beam

Conclusion

Particle representation	→	A 6D vector (position, motion)
Accelerator representation	→	A synchronous particle
Longitudinal dynamics	→	(Phase, Energy) - Hamiltonian
Linear dynamics (6D)	→	Matrix formalism
Periodic structures	→	Courant-Snyder periodic parameters
Beam representation	→	Distribution + momentum (sigma, Twiss)
Matching	→	Twiss = Courant-Snyder parameters
Mismatching	→	Emittance growth